Practice exam 2

- This is a 2 hour exam.
- There are 5 questions in this exam, of which you must answer 4.
- Please show your work. A correct answer will receive little or no credit if adequate work is not shown. In order to receive full credit, the steps leading to the solution must be presented in a clear and coherent manner.
- Indicate below which 4 questions you would like to be graded. If you do not indicate any choice, the first 4 questions will be graded.

  1.
  2.
  3.
  4.
  5.

1. Let $f(z) = \dfrac{\exp(z)}{z^2+1}$ and $g(z) = \dfrac{\exp(\frac{1}{z})}{z^2+1}$.

   Define paths α, β, $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ by

   i. $\alpha(t) = 2\exp(it)$ for all t
   ii. $\beta(t) = i + \exp(it)$ for all t
   iii. $\gamma(t) = -i + \exp(-it)$ for all t.

   (a) Evaluate $\int_\alpha f\, dz$. (5 pts)
   (b) Evaluate $\int_{\alpha+\beta+\gamma} f\, dz$. (10 pts)
   (c) Evaluate $\int_\alpha g\, dz$. (10 pts)

2. Consider the formula $f(z) = \dfrac{\sin(iz)\exp\left(\frac{1}{z-1}\right)}{(z-\pi i)^3(z-2\pi i)}$.

(a) Find the largest open subset Ω of $\mathbb{C}$ where the given formula defines a holomorphic function. (5 pts)

(b) Let $E := \mathbb{C} - \Omega$. For each point $z \in E$, state whether z is a removable singularity, an essential singularity, or a pole. (10 pts)

(c) Compute the order of each zero and each pole. (5 pts)

(d) Find the radius of convergence of the Taylor series of f centered at $z = 2 + i\pi$. (5 pts).

3. For $R \in \mathbb{R}$, $R > 0$, let $\gamma_R = \alpha_R + \beta_R$, where $\alpha,\ \beta : [0,1] \longrightarrow \mathbb{C}$ are defined by $\alpha_R(t) = -R + 2Rt$ and $\beta_R(t) = R\exp(i\pi t)$ for all t. Let $f(z) = \frac{z\exp(iz)}{(z^2+1)(z^2+4)}$ and $g(x) = \frac{x\sin x}{(x^2+1)(x^2+4)}$.

(a) Compute the integral $\int_{\gamma_R} f\,dz$. How does the answer depend on R? (10 pts)

(b) Calculate $\lim_{R\to\infty} \int_{\beta_R} f\,dz$. Explain every step in your calculation. (10 pts)

(c) Use the results of parts (a) and (b) to evaluate $\int_0^\infty g\,dx$. (5 pts).

4. (a) Let $\Omega \subset \mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Let γ be a closed loop in Ω. Which of the following statements is necessarily true? Explain. (10 pts).

A. If $\Omega = \{z \mid 1 < |z| < 2\}$ then $\int_\gamma f\,dz = 0$.

B. If $\Omega = \{z \mid 1 < |z| < 2\}$ and f is analytic, then $\int_\gamma f\,dz = 0$.

C. If $\Omega = \{z \mid Im(z) > 0\}$, and f is analytic and bounded, then f is constant.

D. If $\Omega = \{z \mid Im(z) > 0\}$, then $\int_\gamma f\,dz = 0$.

E. If $\Omega = \{z \mid Im(z) > 0\}$, and f is analytic, then f possesses an analytic antiderivative.

F. If $\Omega = \mathbb{C} - 0$, and f is analytic, then $\int_\gamma f\,dz = 0$.

(b) Solve the wave equation $\triangle u = u_{tt}$, $0 \leq r < 1$, $t > 0$, subject to the initial conditions $u(r, \theta, 0) = 1 - r^2$ and $u_t(r, \theta, 0) = 0$. Here (r, θ) are polar coordinates on $\mathbb{R}^2$. (15 pts).

5. (a) Solve the equation $\triangle u = 0$, $0 < x^2 + y^2 < 1$, $y > 0$, subject to the boundary conditions $u|_{y=0} = 0$ and $u|_{\{x^2+y^2=1\}} = y$. Here (x, y) are cartesian coordinates on $\mathbb{R}^2$. (10 pts)

(b) Let $r \in \mathbb{R}$, $r > 0$, and let R_r be the rectangle in $\mathbb{C}$ with vertices at $-r$, r, $r + i\frac{b}{2a^2}$, $-r + i\frac{b}{2a^2}$, and γ_r the boundary of R_r.

Let $f(z) = exp(-a^2 z^2 + ibz)$ and let $g(x) = exp(-a^2 x^2)\cos(bx)$.

i. Evaluate the integral $\int_{\gamma_r} f\, dz$. (5 pts)

ii. Compute $\lim_{r\to\infty} \int_{\gamma_r} f\, dz$. (5 pts)

iii. Use your answers from (i) and (ii) and the fact that $\int_{-\infty}^{\infty} exp(-a^2x^2)\, dx = \frac{\sqrt{\pi}}{a}$ to evaluate $\int_0^{\infty} g\, dx$. (5 pts)