Problem set 1

Due Wednesday, June 4

1. (a) Show that every solution of the wave equation $u_{tt} = a^2 u_{xx}$ on the real line is of the form $u(x,t) = f(x - at) + g(x + at)$ for some real valued functions f and g.

Hint: Use the change of variables $\xi = x - at$ and $\eta = x + at$ to rewrite the equation as $u_{\xi\eta} = 0$

(b) If $\phi(x) := u(x,0)$ and $\psi(x) := u_t(x,0)$, then show that

$$u(x,t) = \frac{\phi(x-at) + \phi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)dy$$

Hint: (i) Put $t = 0$ in the equation $u(x,t) = f(x-at) + g(x+at)$, and differentiate the resulting equation. (ii) Differentiate the equation $u(x,t) = f(x-at) + g(x+at)$ with respect to t. Put $t = 0$ in the resulting equation. (iii) Solve the resulting linear equations for $f'(x)$ and $g'(x)$. (iv) Integrate to find f and g.

2. Let α be the real valued function whose graph is shown below. Consider a semi-infinte string along the x-axis. The end of the string at $x = 0$ moves freely along a rod which lies perpendicular to the string.

   (a) Draw a movie showing the evolution of the string if the displacement u satisfies $u_{xx} = u_{tt}$ and $u|_{t=0} = \alpha$ and $u_t|_{t=0} \equiv 0$. Be sure to include snapshots for $t = 0$, $t << \delta$, $t = \delta$, $t = 1/2 - \delta$, $1/2 - \delta < t < 1/2 + \delta$, and $t >> 1$.

(b) The same as in part (a), with the initial conditions $u|_{t=0} = 0$ and $u_t|_{t=0} = \alpha$

3. (a) The same as in problem 2(a), with the semi-infinite string replaced by a finite string of length 1 whose ends are *fixed* to walls at $x = 0$ and $x = 1$.

(b) The same as in problem 3(a), except that the end of the string at $x = 1$ is free to move (along a rod perpendicular to the string). The end of the string at $x = 0$ is fixed.

4. Consider a semi-infinite string that undergoes forced oscillations: $u_{xx} = u_{tt}$, $u(0,t) = f(t)$. Can you write down a formula $u(x,t)$ that is analogous the the formula that you derived in $1(b)$? Try the case where $u_t|_{t=0} = 0$ first.

5. Draw a phase portrait of the equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$. Solve the equation for $\mathbf{x}(t)$, subject to the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$