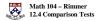
12.4 Comparison Tests



The Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n$, $(a_n \ge 0)$

- (i) if the terms a_n are smaller than the terms b_n of a known convergent series $\sum_{n=1}^{\infty} b_n$ ($b_n \ge 0$), then our series $\sum_{n=1}^{\infty} a_n$ is also convergent.
- (ii) if the terms a_n are larger than the terms b_n of a known **divergent** series $\sum_{n=1}^{\infty} b_n (b_n \ge 0)$, then our series $\sum_{n=1}^{\infty} a_n$ is also **divergent**.

For the series $\sum_{n=1}^{\infty} b_n$, it must be known whether it converges or diverges, so it is usually chosen to be a p-series or a geometric series.

search for the **dominating terms** in both the numerator and the denominator of a_n , choose your b_n to be the ratio of these dominating terms

Math 104 – Rimmer 12.4 Comparison Te

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} \cdot 4^n}$

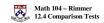
Choose $b_n = \frac{1}{4^n}$ since as n gets large 4^n is much larger than $\sqrt{n+1}$.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4^n}$$
 is a convergent geometric series

$$\frac{1}{\sqrt{n+1}\cdot 4^n} < \frac{1}{4^n} \text{ for } n > 1 \text{ since you are making the}$$

denominator smaller by taking away $\sqrt{n+1}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} \cdot 4^n}$$
 also converges by the Comparison Test



Consider the series $\sum_{n=9}^{\infty} \frac{\sqrt{n}}{n-8}$

Choose $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ since as *n* gets large *n* is much larger than 8.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$
 is a divergent p – series

$$\frac{1}{\sqrt{n}} < \frac{\sqrt{n}}{n-8} \text{ since } n-8 < n$$

$$\Rightarrow \sum_{n=9}^{\infty} \frac{\sqrt{n}}{n-8}$$
 also diverges by the Comparison Test



The inequality $a_n \le b_n$ or $b_n \le a_n$ doesn't need to be satisfied for all values of n. If it doesn't hold for the first few terms but it holds for all n > N for some N, then the comparison test will still work.

Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ Choose $b_n = \frac{1}{n}$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, so it is divergent

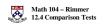
Since $\sum_{n=1}^{\infty} b_n$ is divergent, the inequality should be $\frac{1}{n} \le \frac{\ln n}{n}$

$$\Rightarrow n \le n \ln n \ \Rightarrow \frac{n}{n} \le \frac{n \ln n}{n} \ \Rightarrow 1 \le \ln n \ \Rightarrow e^1 \le e^{\ln n} \ \Rightarrow n > e$$

The inequality doesn't hold for n = 1 or n = 2 but it holds for all $n \ge 3$

The convergence or divergence of the series does not depend on the first two terms. These terms can be subtracted off and we can look at both series starting at n = 3.

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ also diverges by the Comparison Test}$$



Consider the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 1}}$

Choose $b_n = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$ since as *n* gets large *n* the "+8" won't matter.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$
 is a divergent p – series

but the inequality is going the wrong way!

$$\frac{n}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n}} \text{ since } n^{3/2} < \sqrt{n^3+1}$$

The Comparison Test does not apply. \Rightarrow We must use another test.



The Limit Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n$, $(a_n > 0)$ and a known

convergent or divergent series $\sum_{n=1}^{\infty} b_n$, $(b_n > 0)$

If the $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite positive number, then

the series will behave alike, i.e. either both converge or both diverge.

Back to the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$. Choose the same $b_n = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{n}{\sqrt{n^3 + 1}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{n^{3/2}}{\sqrt{n^3 + 1}} = \lim_{n \to \infty} \frac{n^{3/2}}{\sqrt{n^3 + 1}} \cdot \frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n \to \infty} \frac{1}{\sqrt{\frac{n^3}{n^3} + \frac{1}{n^3}}} = 1$$

⇒ the series will behave alike by the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 1}} \text{ also diverges}$$

Consider the series $\sum_{n=1}^{\infty} \frac{1+3^n}{4+2^n}$ Choose $b_n = \frac{3^n}{2^n}$ $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a divergent geometric series w/ $r = \frac{3}{2}$

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \text{ is a divergent}$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1+3^n}{4+2^n}}{\frac{3^n}{2^n}} = \lim_{n \to \infty} \frac{1+3^n}{4+2^n} \cdot \frac{2^n}{3^n} = \lim_{n \to \infty} \frac{2^n+6^n}{4\cdot 3^n+6^n} = \lim_{n \to \infty} \frac{\frac{2^n}{6^n} + \frac{6^n}{6^n}}{\frac{4\cdot 3^n}{6^n} + \frac{6^n}{6^n}}$$

$$= \lim_{n \to \infty} \frac{\left(\frac{1}{3}\right)^n + 1}{4\cdot \left(\frac{1}{2}\right)^n + 1} = \lim_{n \to \infty} \frac{\left(\frac{1}{3}\right)^n - 1}{4\cdot \left(\frac{1}{2}\right)^n - 1} = 1$$

⇒ the series will behave alike by the Limit Comparison Test

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} \frac{1+3^n}{4+2^n} \text{ also diverges}$$

Consider the series
$$\sum_{n=1}^{\infty} \frac{3n+4}{\left(2n+1\right)^3}$$
 Choose $b_n = \frac{n}{n^3} = \frac{1}{n^2}$ Math 104 - Rimmer 12.4 Comparison Test
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a convergent}$$

$$p - \text{series w}/p = 2$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{3n+4}{\left(2n+1\right)^3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{3n+4}{\left(2n+1\right)^3} \cdot \frac{n^2}{1} = \lim_{n \to \infty} \frac{3n^3+4n^2}{\left(2n+1\right)^3} = \frac{3}{8}$$

$$\deg(num.) = \deg.(denom.)$$

$$\lim_{x \to \infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} = \frac{a_m}{b_m}$$

⇒ the series will behave alike by the Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } \Rightarrow \sum_{n=1}^{\infty} \frac{3n+4}{(2n+1)^3} \text{ also converges}$$