Supplementary exam problems for Chapter 12

1. Let $b > a > 1$ be integers. Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} .$$

Hint: use a partial fraction expansion.

Extra problem: Can you sum $1/[(n+a)(n+b)(n+c)]$ the same way?

2. Define p_n and q_n recursively by $p_n = q_n = 1$ and

$$\begin{aligned} p_{n+1} &= 2q_n + p_n \\ q_{n+1} &= 3q_n + p_n . \end{aligned}$$

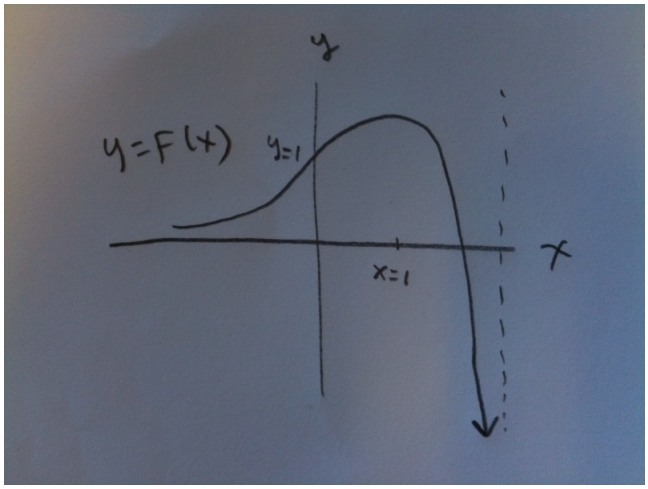
Let $a_n = p_n/q_n$. Prove that a_n converges to a finite limit and evaluate the limit.

3. A superball, when dropped from height h, returns to height $h/(1+h)$. Does such a ball travel a finite distance or an infinite distance if dropped from a positive height and left to bounce forever?

4. Write the series $\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$ in $\sum$ notation. Then determine whether the series converges.

5. A power series for a function $f(x)$ begins $1 + ax + bx^2 + \cdots$. Find the first three terms of the power series for the function $\sqrt{f(x)}$.

6. Estimate the first several terms of the Maclaurin series for the function f whose graph is shown: give numerical estimates for the coefficients of the first two terms and the signs of the next two terms. Also, give a guess as to the radius of convergence.



7. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}$ converges to a function $f(x)$ when $|x|$ is sufficiently small.

   (a) Write the first four terms of this series.

   (b) Find the radius of convergence of the series.

   (c) Find an explicit formula for f in terms of elementary functions (for example $\sin x$, $\sqrt{1 + e^x}$, etc.).

8. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ converges because it is an alternating series whose terms go to zero. Let S denote the sum of this series. Use power series to find an exact expression for S.

9. (a) Use the formula $a_n = f^{(n)}(0)/n!$ to write the first three terms of the Maclaurin series for the function $f(x) = \sqrt{2 - e^x}$.

   (b) Use another method to find these three terms: compute the power series for $2 - e^x$ and use the result of problem 5.

   (c) What is the radius of convergence of this series? Hint: don't use the ratio test – this is very difficult – instead, have a look at the graph of the function.

10. Use your knowledge of the function represented by the power series $\sum_{n=1}^{\infty} nx^{n-1}$ to determine the exact value of the sum

$$\sum_{n=1}^{\infty} \frac{n}{3^{n-1}} .$$