

Bounding the error in the midpoint rule for numerical integration

Let f be a twice differentiable function on $[a, b]$. The midpoint rule with n intervals approximates $\int_a^b f(x) dx$ by

$$S = \frac{b-a}{n} \sum_{j=1}^n f(x_j)$$

where x_j is the midpoint of the interval

$$I_j = \left[a + (j-1) \frac{b-a}{n}, a + j \frac{b-a}{n} \right],$$

which we denote (a_j, b_j) . The error from the midpoint rule on the interval I_j is the discrepancy

$$\Delta_j = \frac{b-a}{n} f(x_j) - \int_{a_j}^{b_j} f(x) dx. \quad (1)$$

We may write $(b-a)f(x_j)/n$ as the integral of a constant: $\int_{a_j}^{b_j} f(x_j) dx$; actually it is more helpful if we include a linear term:

$$\frac{b-a}{n} f(x_j) = \int_{a_j}^{b_j} [f(x_j) + f'(x_j)(x-x_j)] dx; \quad (2)$$

this returns the same value as the constant integral because the linear term integrates to zero. Using (2) we see that

$$\frac{b-a}{n} f(x_j) - \int_{a_j}^{b_j} f(x) dx = \int_{a_j}^{b_j} R_1(x) dx$$

where $R_1(x)$ is the Taylor remainder $f(x) - f(x_j) - f'(x_j)(x-x_j)$. By Taylor's remainder Theorem, $|R_1(x)| \leq M(x-x_j)^2/2$ where M is an upper bound for f'' on I_j . Integrating $(x-x_j)^2$ on I_j gives $(b_j-a_j)^3/12 = (b-a)^3/(12n^3)$, so this is an upper bound for $|\Delta_j|$. Summing this over the n intervals I_j gives a total error of at most

$$\frac{(b-a)^3 M}{24n^2}.$$