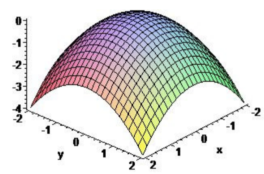
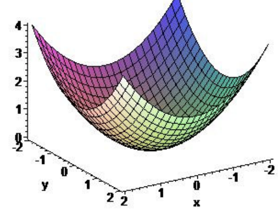
A function of two variables has a **<u>local maximum</u>** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in some region around (a,b).

- outside the region it is possible that the function could be larger



A function of two variables has a **<u>local minimum</u>** at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in some region around (a,b).

- outside the region it is possible that the function could be smaller



A point (a,b) is called a **critical point** of f is one of the following is true :

$i)$ $\nabla f(a,b) = \mathbf{0}$, that is BOTH $f_x(a,b) = 0$ and $f_y(a,b) = 0$

$ii)$ $f_x(a,b)$ and / or $f_y(a,b)$ doesn't exist

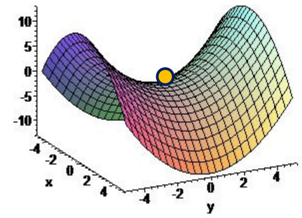
f has a local maximum or local minimum at (a,b) and the first partial derivatives of f exist there $\Longrightarrow$ $f_x(a,b) = 0$ and $f_y(a,b) = 0$ $\Downarrow$ (a,b) is a critical point

f has a local maximum or local minimum at (a,b) $\Longrightarrow$ (a,b) is a critical point

$\not\Longleftarrow$

not all critical points lead to a local maximum or local minimum

$z = \frac{1}{2}\left(y^2 - x^2\right)$



$f(0,0) = 0$

maximum in the direction of the x-axis

minimum in the direction of the y-axis

the graph is in the shape of a saddle

the point (0,0,0) is called a *saddlepoint*

Find all critical points (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$

and

the second partial derivatives are contiuous in some region around (a,b)

Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (f_{xx})(f_{yy}) - (f_{xy})^2$$

Evaluate D at these critical points

$D(a,b) > 0$ → $f_{xx}(a,b) > 0$ → $f(a,b)$ is a **local minimum**

$D(a,b) > 0$ → $f_{xx}(a,b) < 0$ → $f(a,b)$ is a **local maximum**

$D(a,b) < 0$ → $f(a,b)$ is a **saddle point**

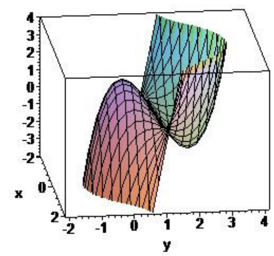
$D(a,b) = 0$ → the test gives no information

$$f(x,y)=3x^2y+y^3-3x^2-3y^2+2$$

$f_x=6xy-6x \qquad f_y=3x^2+3y^2-6y$

$f_x=6x(y-1)$

$f_x=0 \Rightarrow$ either $6x=0$ or $y-1=0$



$(a)\ x=0 \Rightarrow f_y=3y^2-6y=3y(y-2)$

$\Rightarrow y=0$ or $y=2$

$(0,0)$ and $(0,2)$

$(b)\ y=1 \Rightarrow f_y=3x^2+3-6 \ =3(x^2-1)$

$\Rightarrow x=1$ or $x=-1$

$(1,1)$ and $(-1,1)$

$$f(x,y)=3x^2y+y^3-3x^2-3y^2+2$$

$f_x=6xy-6x \qquad f_y=3x^2+3y^2-6y$

$f_{xx}=6y-6 \qquad f_{yy}=6y-6$

$f_{xy}=6x$

$D=(6y-6)^2-(6x)^2$

$D=36\left[(y-1)^2-x^2\right]$

$D(0,0)=36$
$f_{xx}(0,0)=-6$

$D(0,2)=36$
$f_{xx}(0,2)=12$

$D(1,1)=-36$
$f_{xx}(1,1)=$ doesn't matter

$D(-1,1)=-36$
$f_{xx}(-1,1)=$ doesn't matter

	D	f_{xx}	Classification
$(0,0,\mathbf{2})$	>0	<0	**local max.**
$(0,2,\mathbf{-2})$	>0	>0	**local min.**
$(1,1,\mathbf{0})$	<0	$--$	**saddle pt.**
$(-1,1,\mathbf{0})$	<0	$--$	**saddle pt.**

A function of two variables has an **absolute maximum** at (a,b) if $f(x,y) \le f(a,b)$ for all points in the domain of f

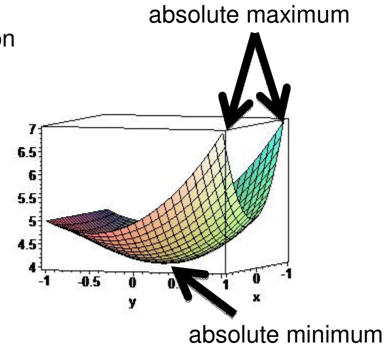
A function of two variables has an **absolute minimum** at (a,b) if $f(x,y) \ge f(a,b)$ for all points in the domain of f

Usually the domain is restricted to some region

$f(x,y) = x^2 + y^2 + x^2 y + 4$

Restricted Domain :

$-1 \le x \le 1$ and $-1 \le y \le 1$



A region in R^2 (for us this will be the xy plane) is called **closed** if it includes its boundary.

A region in R^2 (for us this will be the xy plane) is called **bounded** if it is contained within some disk (in other words a region is bounded if it is finite)

Extreme Value Theorem

$f(x,y)$ is continuous in some closed bounded region S in R^2 $\Longrightarrow$ there are points (a,b) and (c,d) in the region S so that $f(a,b)$ is an absolute maximum and $f(c,d)$ is an absolute minimum

This tells us that the points exist but it doesn't tell us how to find them.

To find the absolute maximum and absolute minimum values of a continuous function f on a closed region S:

1) Find all the critical points of f that lie in the region S
Evaluate the function at each of these points

2) Find all extreme values of f that lie on the boundary. (This turns into a Calc I problem)

3) The largest and smallest of the values found in steps 1 and 2 are the absolute maximum value and absolute minimum value of the function f

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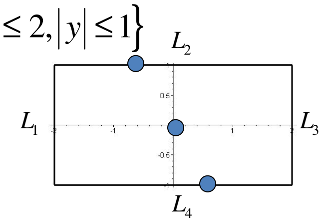
Find the absolute maximum and absolute minimum values of

$f(x,y)=x^2+xy$ on the region $S:\{(x,y)\,|\,|x|\le 2, |y|\le 1\}$



1) Find possible critical pts. inside the region

$f(x,y)=x^2+xy$

$f_x=2x+y\overset{set}{=}0 \qquad f_y=x\overset{set}{=}0$

Both have to be true at the same time
plugging in $x=0$ into $f_x \Rightarrow y=0$

Critical pt.
$(0,0)$

2) Find all extreme values of f that lie on the boundary.

a) $L_1: x=-2 \quad f(-2,y)=(-2)^2+(-2)y \Rightarrow f$ on $L_1=-2y+4$
f' on $L_1=-2\neq 0$ no extreme points on L_1

b) $L_2: y=1 \quad f(x,1)=x^2+x \Rightarrow f$ on $L_2=x^2+x$
f' on $L_2=2x+1\overset{set}{=}0 \Rightarrow x=\frac{-1}{2}$ $\left(\frac{-1}{2},1\right)$

c) $L_3: x=2 \quad f(2,y)=2^2+2y \Rightarrow f$ on $L_3=2y+4$
f' on $L_3=2\neq 0$ no extreme points on L_3

d) $L_4: y=-1 \quad f(x,-1)=x^2-x \Rightarrow f$ on $L_4=x^2-x$
f' on $L_4=2x-1 \Rightarrow x=\frac{1}{2}$ $\left(\frac{1}{2},-1\right)$

$f(0,0)=0$ absolute maximum

$f\left(\frac{-1}{2},1\right)=\frac{-1}{4}$ absolute minimum

$f\left(\frac{-1}{2},1\right)=\frac{-1}{4}$ absolute minimum