The Inverse of a Matrix

A square $(n\times n)$ matrix A is **invertible** if there exists an $n\times n$ matrix B such that $AB = BA = I_{n\times n}$.

$I_{n\times n}$ is the $n\times n$ identity matrix.

B is called the (multiplicative) inverse of A.

The symbol used for B is A^{-1}.

A^{-1} is unique.

Not all matrices are invertible. A matrix that is not invertible is called **singular**.

example:

$$A=\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}$$

$$A^{-1}=\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$$

$$AA^{-1}=\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}\begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$$

$$AA^{-1}=\begin{pmatrix}2(3)+5(-1) & 2(-5)+5(2)\\ 1(3)+3(-1) & 1(-5)+3(2)\end{pmatrix}$$

$$AA^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad AA^{-1}=I_{2\times 2}$$

You only need to check one direction since:

$AB = I$

$\underbrace{(AB)}_{I}A = A$ (mult. on rt. by A)

$A(BA) = A$ (mult. is assoc.)

$\Rightarrow BA = I$ (since $AI = A$)

Finding the Inverse of a 2 X 2 Matrix

$$\text{Let } A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

❶ Switch a and d

❷ Negate b and c

❸ Calculate $D = ad - bc$

❹ Divide every entry by D.

$$A^{-1}=\begin{pmatrix}d/D & -b/D\\ -c/D & a/D\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \qquad A^{-1}=\begin{pmatrix}4/_{-2} & -2/_{-2}\\ -3/_{-2} & 1/_{-2}\end{pmatrix}=\begin{pmatrix}-2 & 1\\ 3/2 & -1/2\end{pmatrix}$$

$D = ad - bc$

$D = 1(4) - 2(3)$

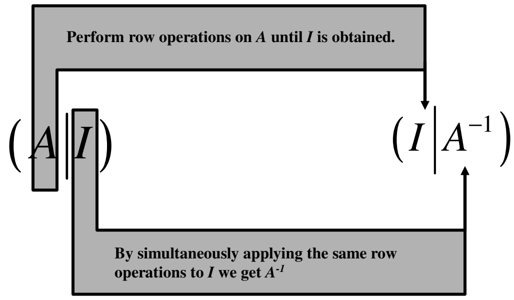
$D = -2$

$$AA^{-1}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}-2 & 1\\ 3/2 & -1/2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Finding the Inverse of a 3 X 3 (or larger) Matrix (Method 1)

Let A be $n\times n$. Adjoin (attach) the $n\times n$ Identity matrix.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix.}}_{A}\left|\underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{matrix}}_{I}\right)$$



The 3 Elementary Row Operations :

a) Multiply a row by a number (nonzero)

b) Switch rows

c) Add a multiple of $\underbrace{\text{one row}}_{\text{Row that is not changing}}$ to $\underbrace{\text{another row}}_{\text{Row you want to replace}}$

Let $A=\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$. Find A^{-1}.

$$\left(\underbrace{\begin{matrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{matrix}}_{A}\right|\left.\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) 2R_1 \Rightarrow \left(\begin{array}{ccc|ccc} 4 & 0 & 2 & 2 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) R_3+R_1=NewR_1 \Rightarrow \left(\begin{array}{ccc|ccc} -1 & 5 & 8 & 2 & 0 & 1 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) -R_1 \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \begin{matrix} 2R_1+R_2=NewR_2 \Rightarrow \\ 5R_1+R_3=NewR_3 \Rightarrow \end{matrix}$$

$$\Rightarrow \begin{array}{r|rrr|rrr} 2R_1 & 2 & -10 & -16 & -4 & 0 & -2 \\ +\ R_2 & -2 & 3 & 4 & 0 & 1 & 0 \\ \hline NewR_2 & 0 & -7 & -12 & -4 & 1 & -2 \end{array} \qquad \Rightarrow \begin{array}{r|rrr|rrr} 5R_1 & 5 & -25 & -40 & -10 & 0 & -5 \\ +\ R_3 & -5 & 5 & 6 & 0 & 0 & 1 \\ \hline NewR_3 & 0 & -20 & -34 & -10 & 0 & -4 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ 0 & -7 & -12 & -4 & 1 & -2 \\ 0 & -20 & -34 & -10 & 0 & -4 \end{array}\right) -3R_2 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ 0 & 21 & 36 & 12 & -3 & 6 \\ 0 & -20 & -34 & -10 & 0 & -4 \end{array}\right) R_3+R_2=NewR_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & -20 & -34 & -10 & 0 & -4 \end{array}\right)$$

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$. Find A^{-1}. (continued)

$$\left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & -20 & -34 & -10 & 0 & -4 \end{array}\right) \begin{array}{l} 5R_2 + R_1 = NewR_1 \Rightarrow \\ \\ 20R_2 + R_3 = NewR_3 \Rightarrow \end{array}$$

$$\Rightarrow \begin{array}{r|ccc|ccc} 5R_2 & 0 & 5 & 10 & 10 & -15 & 10 \\ + \ R_1 & 1 & -5 & -8 & -2 & 0 & -1 \\ \hline NewR_1 & 1 & 0 & 2 & 8 & -15 & 9 \end{array} \qquad \Rightarrow \begin{array}{r|ccc|ccc} 20R_2 & 0 & 20 & 40 & 40 & -60 & 40 \\ + \ R_3 & 0 & -20 & -34 & -10 & 0 & -4 \\ \hline NewR_3 & 0 & 0 & 6 & 30 & -60 & 36 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 8 & -15 & 9 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & 0 & 6 & 30 & -60 & 36 \end{array}\right) \tfrac{1}{6}R_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 8 & -15 & 9 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right) \begin{array}{l} -2R_3 + R_1 = NewR_1 \\ -2R_3 + R_2 = NewR_2 \\ \\ \end{array}$$

$$\Rightarrow \begin{array}{r|ccc|ccc} -2R_3 & 0 & 0 & -2 & -10 & 20 & -12 \\ + \ R_1 & 1 & 0 & 2 & 8 & -15 & 9 \\ \hline NewR_1 & 1 & 0 & 0 & -2 & 5 & -3 \end{array} \qquad \Rightarrow \begin{array}{r|ccc|ccc} -2R_3 & 0 & 0 & -2 & -10 & 20 & -12 \\ + \ R_2 & 0 & 1 & 2 & 2 & -3 & 2 \\ \hline NewR_2 & 0 & 1 & 0 & -8 & 17 & -10 \end{array}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} \underbrace{\begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}_{A^{-1}} \qquad A^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix} \qquad \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I}$$

Let $A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$. Find A^{-1}.

$$\underbrace{\begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} \begin{array}{l} \\ -2R_1 + R_2 = NewR_2 \Rightarrow \\ -6R_1 + R_3 = NewR_3 \Rightarrow \end{array}$$

$$\Rightarrow \begin{array}{r|ccc|ccc} -2R_1 & -2 & 2 & 4 & -2 & 0 & 0 \\ + \ R_2 & 2 & 4 & 5 & 0 & 1 & 0 \\ \hline NewR_2 & 0 & 6 & 9 & -2 & 1 & 0 \end{array} \qquad \Rightarrow \begin{array}{r|ccc|ccc} -6R_1 & -6 & 6 & 12 & -6 & 0 & 0 \\ + \ R_3 & 6 & 0 & -3 & 0 & 0 & 1 \\ \hline NewR_3 & 0 & 6 & 9 & -6 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 6 & 9 & -6 & 0 & 1 \end{array}\right) -R_2 + R_3 = NewR_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right)$$

$$\Rightarrow \begin{array}{r|ccc|ccc} -R_2 & 0 & -6 & -9 & 2 & -1 & 0 \\ + \ R_3 & 0 & 6 & 9 & -6 & 0 & 1 \\ \hline NewR_3 & 0 & 0 & 0 & -4 & -1 & 1 \end{array}$$

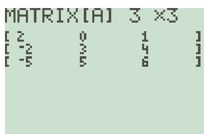
Further reduction will always yield another matrix with a row of zeros. This matrix can never be turned into I.

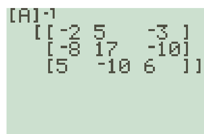
$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$ is not invertible.

A is **singular**.

Properties of the Inverse

- $\left(A^{-1}\right)^{-1} = A$
- $\left(A^{k}\right)^{-1} = \left(A^{-1}\right)^{k}, \quad k > 0$
- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$





Use the 

key

Use the Inverse to Solve a System of Equations

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 1 \\ -2x_1 & + & 3x_2 & + & 4x_3 & = & -2 \\ -5x_1 & + & 5x_2 & + & 6x_3 & = & 3 \end{array} \Rightarrow \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}_{\mathbf{b}}$$

$A\vec{x} = \vec{b}$ and A is invertible

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Earlier we found $A^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}_{\mathbf{b}}$$

To solve the system for x, just find the inverse, and multiply it by b (in the order $A^{-1}b$)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -21 \\ -72 \\ 43 \end{pmatrix}$$