Sections 10.1/10.2
Homogeneous Linear System of Ordinary Differential Equations

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

represents a system of homogeneous first order linear differential equations with constant coefficients

$$\underbrace{\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}}_{\mathbf{X}'} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}}_{\mathbf{X}}$$

example:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} x_1' &= 2x_1 + 4x_2 \\ x_2' &= -x_1 + 6x_2 \end{aligned}$$

Assume the solution vector has the form:

$\mathbf{X} = \mathbf{V}e^{rt}$ ($\mathbf{X}$ and $\mathbf{V}$ vectors)

$\mathbf{X}' = r\mathbf{V}e^{rt}$

$\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{V}e^{rt}$

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \Rightarrow r\mathbf{V}e^{rt} = \mathbf{A}\mathbf{V}e^{rt}$$

$$r\mathbf{V} = \mathbf{A}\mathbf{V}$$

$$\mathbf{A}\mathbf{V} = r\mathbf{V}$$

$(\text{matrix})(\text{vector}) = (\text{constant})(\text{same vector})$

$r = \lambda$ **eigenvalue**

$\mathbf{V}$ **eigenvector**

$$\mathbf{X} = \mathbf{V}e^{\lambda t}$$

The solutions vary based on the nature of the eigenvalues

Distinct Real Eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$: n distinct eigenvalues with corresponding eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$

$$\mathbf{X} = c_1\mathbf{V}_1 e^{\lambda_1 t} + c_2\mathbf{V}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{V}_n e^{\lambda_n t}$$

Repeated Eigenvalues (with *n* linearly independent eigenvectors)

Say $n = 2, \underbrace{\lambda_1 = \lambda_2}_{\text{call it } \lambda}$ with 2 linearly independent eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$

$$\mathbf{X} = c_1\mathbf{V}_1 e^{\lambda t} + c_2\mathbf{V}_2 e^{\lambda t}$$

Repeated Eigenvalues (with < *n* linearly independent eigenvectors)

Say $n = 2, \underbrace{\lambda_1 = \lambda_2}_{\text{call it } \lambda}$ with only one eigenvector $\mathbf{V}_1$

$$\mathbf{X}_1 = \mathbf{V}_1 e^{\lambda t} \qquad \mathbf{X}_2 = \mathbf{V}_1 t e^{\lambda t} + \mathbf{V}_2 e^{\lambda t}$$

$\mathbf{V}_2$ found by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_2 = \mathbf{V}_1$

Solution

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1\mathbf{V}_1 e^{\lambda t} + c_2\left[\mathbf{V}_1 t e^{\lambda t} + \mathbf{V}_2 e^{\lambda t}\right]$$

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

represents a system of first order linear differential equations where:

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \quad \underbrace{\mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}}_{\text{initial condition}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1' = 2x_1 + 4x_2$$
$$x_2' = -x_1 + 6x_2$$

1) Find the eigenvalues and eigenvectors of ***A*.**

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \qquad \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(6-\lambda) + 4 = 0$$
$$12 - 8\lambda + \lambda^2 + 4 = 0$$
$$\lambda^2 - 8\lambda + 16 = 0$$

or use:

2×2 Characteristic Equation Shortcut

$$\lambda^2 - (trace(A))\lambda + \det(A) = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

Repeated eigenvalue

$$\boxed{\lambda_1 = \lambda_2 = 4}$$

Solve: $(\mathbf{A} - 4\mathbf{I})\mathbf{V}_1 = \mathbf{0}$

$$\left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right) \tfrac{1}{2}R_1 \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-v_1 + 2v_2 = 0$ $\quad v_1 = 2v_2$

v_2 is free $\quad$ let $v_2 = 1$

$$\boxed{\mathbf{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Eigenvalue with multiplicity 2 and only 1 eigenvector

Solution # 1

$\mathbf{X}_1 = \mathbf{V}_1 e^{\lambda_1 t}$ $\quad\quad$ $\mathbf{V}_1$ found by solving $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{V}_1 = \mathbf{0}$

Solution # 2

$\mathbf{X}_2 = \mathbf{V}_1 t e^{\lambda_1 t} + \mathbf{V}_2 e^{\lambda_1 t}$ $\quad\quad$ $\mathbf{V}_2$ found by solving $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{V}_2 = \mathbf{V}_1$

Full Solution

$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$

Solve: $(\mathbf{A}-4\mathbf{I})\mathbf{V}_2 = \mathbf{V}_1$

$$\left(\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right) \tfrac{1}{2}R_1 \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 1 \\ -1 & 2 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} -v_1 + 2v_2 = 1 \\ v_2 \text{ is free} \end{array} \quad \begin{array}{l} v_1 = 2v_2 - 1 \\ \text{let } v_2 = 1 \end{array} \quad \boxed{\mathbf{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\mathbf{X}_1 = \mathbf{V}_1 e^{\lambda_1 t} \Rightarrow \mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$$

$$\mathbf{X}_2 = \mathbf{V}_1 t e^{\lambda_1 t} + \mathbf{V}_2 e^{\lambda_1 t} \Rightarrow \mathbf{X}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

Full Solution

$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$

$$\boxed{\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-4t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \right]}$$

$\mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$ initial condition

$$\mathbf{X}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 & c_2 \\ c_1 & c_2 \end{pmatrix} \overset{\text{set}}{=} \begin{pmatrix} -1 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 1 & 6 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{cc|c} 1 & 1 & 6 \\ 2 & 1 & -1 \end{array}\right) R_2 - 2R_1 \left(\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -1 & -13 \end{array}\right) \begin{array}{l} c_1 + c_2 = 6 \\ c_2 = 13 \end{array} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -7 \\ 13 \end{pmatrix}$$

$$\mathbf{X} = -7\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}\right]$$

$$\mathbf{X} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} e^{4t} + \begin{pmatrix} 26 \\ 13 \end{pmatrix} te^{4t}$$

$$\boxed{\mathbf{X} = \begin{pmatrix} -e^{4t} + 26te^{4t} \\ 6e^{4t} + 13te^{4t} \end{pmatrix}}$$

<u>Check:</u>

$$\mathbf{X}' = \begin{pmatrix} -4e^{4t} + 26e^{4t} + 104te^{4t} \\ 24e^{4t} + 13e^{4t} + 52te^{-4t} \end{pmatrix} = \begin{pmatrix} 22 & 104 \\ 37 & 52 \end{pmatrix} \begin{pmatrix} e^{4t} \\ te^{4t} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} -e^{4t} + 26te^{4t} \\ 6e^{4t} + 13te^{4t} \end{pmatrix} = \begin{pmatrix} -1 & 26 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} e^{4t} \\ te^{4t} \end{pmatrix}$$

$$\mathbf{X}' = \mathbf{AX} \quad \checkmark$$

$$\mathbf{AX} = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} -1 & 26 \\ 6 & 13 \end{pmatrix} \begin{pmatrix} e^{4t} \\ te^{4t} \end{pmatrix} = \begin{pmatrix} 22 & 104 \\ 37 & 52 \end{pmatrix} \begin{pmatrix} e^{4t} \\ te^{4t} \end{pmatrix}$$