Math 240 – Rimmer Name **ANSWER KEY**

Midterm # 3 – Fall 2007

December 4, 2007

1.

Evaluate $\iint_S (6x-3y)\,dA$

where S is the region bounded by the lines

$2x-y=1 \quad 2x-y=3$

$x+y=1 \quad x+y=2$

A) 2 B) 3 C) 12 D) 4 E) 6

$$u=2x-y \quad 1\le u\le 3$$
$$v=x+y \quad 1\le v\le 2$$

$$\begin{array}{l} u=2x-y \\ \underline{v=x+y} \\ u+v=3x \end{array} \Rightarrow x=\frac{1}{3}(u+v)\Rightarrow 6x=2u+2v$$

$$v=\frac{1}{3}(u+v)+y\Rightarrow y=\frac{2}{3}v-\frac{1}{3}u\Rightarrow -3y=u-2v \qquad \Rightarrow 6x-3y=3u$$

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix}=\frac{2}{9}+\frac{1}{9}=\frac{1}{3}$$

$$\iint_S (6x-3y)\,dA=\int\int 3u\left|J(u,v)\right|dA=\int_1^2\int_1^3 u\,du\,dv=\int_1^3 u\,du\cdot\int_1^2 dv= \boxed{4}$$

2.
Solve the homogeneous differential equation

$$\left(1+\frac{y}{x}\right)dx-dy=0 \text{ with } y(1)=0$$

Find $f(e)$.

A) e B) 1 C) $\frac{1}{e}$ D) $\frac{1}{e^2}$ E) none of the above

$$y=ux$$
$$dy=udx+xdu$$
$$\frac{y}{x}=u$$

$$\left(1+\frac{y}{x}\right)dx-dy=0 \Rightarrow (1+u)dx-(udx+xdu)=0$$
$$(1\cancel{+u-u})dx-xdu=0$$
$$dx=xdu$$
$$\frac{1}{x}dx=du$$
$$u=\ln x+C$$
$$\frac{y}{x}=\ln x+C$$
$$y=x\ln x+Cx$$
$$y(1)=0 \Rightarrow 0=1\ln 1+C \Rightarrow C=0$$
$$y=x\ln x$$
$$y(e)=e\ln e=\boxed{e}$$

3.

Solve the differential equation

$y' + y = y^3$

with $y(0) = \frac{1}{3}$.

A) $\frac{1}{\sqrt{5e^{-2x}+4}}$ B) $\frac{1}{\sqrt{3e^{-2x}+6}}$ C) $\frac{1}{\sqrt{8e^{-2x}+1}}$ D) $\frac{1}{\sqrt{7e^{-2x}+2}}$ E) none of the above

<u>Bernoulli equation with $n = 3$</u>

$u = y^{-2}$

$y = u^{-1/2}$ and $y^3 = u^{-3/2}$

$$\frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx}$$

$$y' + y = y^3 \Rightarrow -\frac{1}{2}u^{-3/2}\frac{du}{dx} + u^{-1/2} = u^{-3/2}$$

Multiply by $-2u^{3/2}$

$u' - 2u = -2 \Leftarrow$ Linear

integrating factor $\mu = e^{\int P(x)dx} = e^{-2x}$

Mult. both sides by μ:

$$\underbrace{e^{-2x}u' - 2ue^{-2x}} = -2e^{-2x}$$

$$\left(e^{-2x}u\right)' = -2e^{-2x}$$

Integrate both sides

$$e^{-2x}u = -2\int e^{-2x}dx$$

$$e^{-2x}u = e^{-2x} + C$$

Solve for u

$$u = 1 + Ce^{2x}$$

$$y^{-2} = 1 + Ce^{2x}$$

$$y = \left(1 + Ce^{2x}\right)^{-1/2}$$

$$y = \frac{1}{\sqrt{1+Ce^{2x}}} \quad y(0) = 1/3 \Rightarrow \frac{1}{3} = \frac{1}{\sqrt{1+C}} \Rightarrow C = 8$$

$$\boxed{y = \frac{1}{\sqrt{1+8e^{2x}}}}$$

4. Solve

$$x^2y''-xy'+2y=0 \qquad y(1)=-1, y'(1)=-1$$

Find $y\left(e^{\pi}\right)$.

A) $-e^{\pi}$ B) e^{π} C) 0 D) $-\pi$ E) π

Cauchy-Euler Equation $a=1, b=-1, c=2$

Auxillary Equation $\Rightarrow x^r\left(ar^2+(b-a)r+c\right)=0$

$$r^2-2r+2=0$$

$$r=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm 2i}{2}=1\pm i$$

• <u>complex roots $r_1=\alpha+\beta i, r_2=\alpha-\beta i$</u>

$$y=x^{\alpha}\left(c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right)$$

$$y=x\left(c_1\cos(\ln x)+c_2\sin(\ln x)\right)$$

$$y'=\left(c_1\cos(\ln x)+c_2\sin(\ln x)\right)+x\left(-c_1\sin(\ln x)\cdot\frac{1}{x}+c_2\cos(\ln x)\cdot\frac{1}{x}\right)$$

$$y'=\left((c_1+c_2)\cos(\ln x)+(c_2-c_1)\sin(\ln x)\right)$$

$$y(1)=-1\Rightarrow -1=1\left(c_1\underbrace{\cos(\underbrace{\ln 1}_{0})}_{1}+c_2\underbrace{\sin(\underbrace{\ln 1}_{0})}_{0}\right)\Rightarrow c_1=-1$$

$y'(1)=-1\Rightarrow c_1+c_2=-1$ but $c_1=-1\Rightarrow c_2=0$

$$y=-x\cos(\ln x)$$

$$y\left(e^{\pi}\right)=-e^{\pi}\cos\left(\ln e^{\pi}\right)=-e^{\pi}\cos(\pi)=\boxed{e^{\pi}}$$

5. A spring with a **mass** of 2 kg. attached has damping constant 16. A force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. If it starts at equilibrium position with a downward velocity of 2.4 m/s, how much time passes until the spring reaches equilibrium for the first time?

A) $\frac{\pi}{4}$ B) $\frac{\pi}{2}$ C) π D) 2π E) $\frac{\pi}{8}$

$m = 2, \beta = 16$

$F = ks \Rightarrow 12.8 = 0.2k \Rightarrow k = 64$

$x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$

$x'' + 8x' + 32x = 0$

Let $2\lambda = \frac{\beta}{m} \Rightarrow \lambda = 4$, and $\omega^2 = \frac{k}{m} \Rightarrow \omega^2 = 32$

$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2} \Rightarrow r = -4 \pm \sqrt{16-32} \Rightarrow r = -4 \pm 4i$

$x(t) = e^{-\lambda t}\left(c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\ t\right) + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\ t\right)\right)$

$x(t) = e^{-4t}\left(c_1 \cos(4\,t) + c_2 \sin(4\,t)\right)$

starts at equilibrium position $\Rightarrow x(0) = 0 \Rightarrow c_1 = 0$

$x(t) = c_2 e^{-4t} \sin(4\,t)$

starts with a downward velocity of 2.4 m/s $\Rightarrow x'(0) = 2.4 \Rightarrow c_1 = 0$

$x'(t) = -4c_2 e^{-4t} \sin(4\,t) + 4c_2 e^{-4t} \cos(4\,t)$

$x'(0) = 4c_2 = 2.4 \Rightarrow 0.6$

$x(t) = 0.6e^{-4t} \sin(4\,t)$

spring reaches equilibrium for the first time $\Rightarrow x(t) = 0, t > 0$

$0.6e^{-4t} \sin(4\,t) = 0 \Rightarrow \sin(4\,t) = 0 \Rightarrow 4t = \pi \Rightarrow$

$$\boxed{t = \frac{\pi}{4}}$$

6. Let $f(t) = L^{-1}\left\{\dfrac{2s-3}{s^2+2s+10}\right\}$. Find $f\left(\dfrac{\pi}{6}\right)$

A) $\dfrac{-5}{3}e^{-\pi/6}$ B) $2e^{-\pi/6}$ C) $\left(\sqrt{3}-\dfrac{5}{6}\right)e^{-\pi/6}$ D) $\dfrac{5}{6}e^{-\pi/6}$ E) none of the above

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2+9} = \frac{2s}{(s+1)^2+9} - \frac{3}{(s+1)^2+9}$$

$$= \frac{2s+2-2}{(s+1)^2+9} - \frac{3}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9} = 2\frac{(s+1)}{(s+1)^2+9} - \frac{5}{3}\frac{5\cdot\frac{3}{5}}{(s+1)^2+9}$$

$$f(t) = L^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\} = L^{-1}\left\{2\frac{(s+1)}{(s+1)^2+9} - \frac{5}{3}\frac{5\cdot\frac{3}{5}}{(s+1)^2+9}\right\}$$

$$f(t) = 2L^{-1}\left\{\frac{(s+1)}{(s+1)^2+9}\right\} - \frac{5}{3}L^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$f(t) = 2L^{-1}\left\{\frac{(s+1)}{(s+1)^2+9}\right\} - \frac{5}{3}L^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$f(t) = e^{-t}\left(2\cos(3t) - \frac{5}{3}\sin(3t)\right)$$

$$f\left(\frac{\pi}{6}\right) = e^{-\pi/6}\left(2\cos\left(\frac{\pi}{2}\right) - \frac{5}{3}\sin\left(\frac{\pi}{2}\right)\right) = \boxed{-\frac{5}{3}e^{-\pi/6}}$$

7. Find the Laplace Transform of

$$f(t) = \frac{1}{2} t e^{-2t} \sin t$$

by any means other than the definition.

A) $\frac{s+4}{s^2+4s+5}$ B) $\frac{s+2}{\left(s^2+4s+5\right)^2}$ C) $\frac{2s+4}{\left(s^2+4s+5\right)^2}$ D) $\frac{s+2}{s^2-4s-5}$ E) none of the above

$$L\{f(t)\} = \frac{1}{2} L\{te^{-2t} \sin t\} = -\frac{1}{2} \frac{d}{dt}\left[L\{e^{-2t} \sin t\}\right]$$

$$= -\frac{1}{2} \frac{d}{dt}\left[\frac{1}{(s+2)^2+1}\right] = -\frac{1}{2} \frac{d}{dt}\left[\frac{1}{s^2+4s+5}\right] = -\frac{1}{2} \frac{d}{dt}\left[\left(s^2+4s+5\right)^{-1}\right]$$

$$= -\frac{1}{2}\left[-\left(s^2+4s+5\right)^{-2} (2s+4)\right] = \boxed{\frac{s+2}{\left(s^2+4s+5\right)^2}}$$

8. Derive the Laplace transform of

$$f(t) = t\cosh(at)$$

You should find a theorem or property more useful than the definition. "I read it off my table" will receive 3 points.

A) $\frac{-2as}{\left(s^2-a^2\right)^2}$ B) $\frac{2as}{\left(s^2-a^2\right)^2}$ C) $\frac{-\left(s^2+a^2\right)}{\left(s^2-a^2\right)^2}$ D) $\frac{s^2+a^2}{\left(s^2-a^2\right)^2}$ E) none of the above

$$L\{f(t)\} = L\{t\cosh(at)\} = -\frac{d}{dt}L\{\cosh(at)\}$$

$$= -\frac{d}{dt}\left[\frac{s}{s^2-a^2}\right] = -\left[\frac{1\cdot\left(s^2-a^2\right)-s\cdot 2s}{\left(s^2-a^2\right)^2}\right] = -\left[\frac{-s^2-a^2}{\left(s^2-a^2\right)^2}\right] = \boxed{\frac{s^2+a^2}{\left(s^2-a^2\right)^2}}$$

9. Solve $y''-2y'+2y=e^{-2t} \qquad y(0)=0, y'(0)=0$

Using Laplace transforms. Find $y(2\pi)$.

A) $\frac{1}{5}e^{2\pi}$ B) $4e^{2\pi}$ C) $\frac{4}{5}e^{2\pi}$ D) $\frac{e^{2\pi}-e^{-4\pi}}{5}$ E) none of the above

$$L\{y''-2y'+2y\}=L\{e^{-2t}\} \quad y(0)=0, y'(0)=0$$

$$L\{y''\}-2L\{y'\}+2L\{y\}=\frac{1}{s+2}$$

$$s^2L\{y\}-sy(0)-y'(0)-2\left[sL\{y\}-y(0)\right]+2L\{y\}=\frac{1}{s+2}$$

$$\left(s^2-2s+2\right)L\{y\}=\frac{1}{s+2}$$

$$L\{y\}=\frac{1}{(s+2)\left(s^2-2s+2\right)}=\frac{A}{s+2}+\frac{Bs+C}{s^2-2s+2}$$

$$\Rightarrow A\left(s^2-2s+2\right)+(Bs+C)(s+2)=1$$

$$(A+B)s^2+(-2A+2B+C)s+(2A+2C)=1$$

$$A+B=0 \Rightarrow B=-A$$

$$2A+2C=1 \Rightarrow 2C=1-2A \Rightarrow C=\frac{1}{2}-A$$

$$-2A+2B+C=0 \Rightarrow -2A-2A+\frac{1}{2}-A=0 \Rightarrow 5A=\frac{1}{2} \Rightarrow A=\frac{1}{10}$$

$$\Rightarrow B=-\frac{1}{10} \text{ and } C=\frac{1}{2}-\frac{1}{10}=\frac{2}{5}$$

$$L\{y\}=\frac{1}{10}\left(\frac{1}{s+2}\right)+\frac{\frac{1}{10}s+\frac{2}{5}}{s^2-2s+2}=\frac{1}{10}\left(\frac{1}{s+2}\right)+\frac{\frac{1}{10}s}{(s-1)^2+1}+\frac{\frac{2}{5}}{(s-1)^2+1}$$

$$L\{y\}=\frac{1}{10}\left(\frac{1}{s+2}\right)+\frac{-\frac{1}{10}s+\frac{1}{10}-\frac{1}{10}}{(s-1)^2+1}+\frac{\frac{2}{5}}{(s-1)^2+1}$$

$$L\{y\}=\frac{1}{10}\left(\frac{1}{s+2}\right)-\frac{1}{10}\left(\frac{s-1}{(s-1)^2+1}\right)+\frac{\frac{2}{5}-\frac{1}{10}}{(s-1)^2+1}$$

$$L\{y\}=\frac{1}{10}\left(\frac{1}{s+2}\right)-\frac{1}{10}\left(\frac{s-1}{(s-1)^2+1}\right)+\frac{3}{10}\left(\frac{1}{(s-1)^2+1}\right)$$

$$\Rightarrow y=\frac{1}{10}e^{-2t}+e^{-t}\left(-\frac{1}{10}\cos t+\frac{3}{10}\sin t\right)$$

$$y(\pi)=\frac{1}{10}e^{-4\pi}+e^{-2\pi}\left(-\frac{1}{10}\right)= \boxed{\frac{1}{10}\left(e^{-4\pi}-e^{-2\pi}\right)}$$

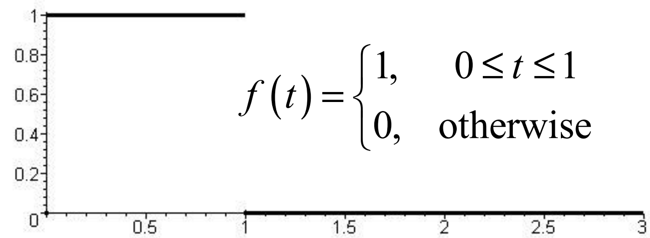
10.

$$x'' - x = f(t)$$

with $x(0)=0, x'(0)=0$

Find the value of the Laplace transform of x evaluated at $s=2$.

(no need to find the function $x(t)$)



A) $\frac{1}{2}\left(1-e^{-2}\right)$ B) $\frac{1}{6}\left(1-e^{-2}\right)$ C) $\frac{8}{3}\left(1-e^{-1/2}\right)$ D) $\frac{1}{3}\left(1-e^{-2}\right)$

E) none of the above

$$L\{x'-x\}=L\{f(t)\}$$

$$L\{x'\}-L\{x\}=L\{f(t)\}$$

$$s^2L\{x\}-s\,\underset{0}{x(0)}-\underset{0}{x'(0)}-L\{x\}=L\{f(t)\}$$

$$\left(s^2-1\right)L\{x\}=L\{f(t)\}$$

$$L\{x\}=\frac{L\{f(t)\}}{\left(s^2-1\right)}$$

$$L\{x\}=\frac{1-e^{-s}}{s\left(s^2-1\right)}$$

$$L\{x\}\text{ at } s=2\Rightarrow\frac{1-e^{-2}}{2(4-1)}=\boxed{\frac{1}{6}\left(1-e^{-2}\right)}$$

$$f(t)=\begin{cases}1, & 0\le t\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$f(t)=\begin{cases}g(t), & 0\le t\le a\\ h(t), & \text{otherwise}\end{cases}\Rightarrow f(t)=g(t)+\left[h(t)-g(t)\right]U(t-a)$$

$$f(t)=1-U(t-1)$$

$$L\{f(t)\}=L\{1-U(t-1)\}=L\{1\}-L\{U(t-1)\}$$

$$L\{f(t)\}=\frac{1-e^{-s}}{s}$$