Section 2.5
Solving a first order O.D.E. by Substitution

$f(x, y)$ is a **homogeneous function** of degree α if $f(tx, ty) = t^{\alpha} f(x, y)$

$(ex.1)$ $g(x, y) = y^2 - x^2$

$$g(tx, ty) = (ty)^2 - (tx)^2$$

$$g(tx, ty) = t^2 \left(y^2 - x^2\right)$$

$$g(tx, ty) = t^2 g(x, y)$$

$(ex.2)$ $h(x, y) = x + \sqrt{xy}$

$$h(tx, ty) = tx + \sqrt{(tx)(ty)}$$

$$h(tx, ty) = tx + t\sqrt{xy} \quad = t\left(x + \sqrt{xy}\right)$$

$$h(tx, ty) = th(x, y)$$

The differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is **homogeneous** if M and N are homogeneous functions of the same degree.

The substitutions $y = ux$ or $x = vy$ will convert the homogeneous equation into a separable equation

WHY ?

$M(x,y)\,dx + N(x,y)\,dy = 0$

Let $M(x,y)$ and $N(x,y)$ be homogeneous of dergree α

$M(tx,ty) = t^{\alpha}M(x,y)$ and $N(tx,ty) = t^{\alpha}N(x,y)$

Let $t = \dfrac{1}{x}$,

$M\left(1,\tfrac{y}{x}\right) = \dfrac{1}{x^{\alpha}}M(x,y)$ and $N\left(1,\tfrac{y}{x}\right) = \dfrac{1}{x^{\alpha}}N(x,y)$

$M(x,y) = x^{\alpha}M\left(1,\tfrac{y}{x}\right)$ and $N(x,y) = x^{\alpha}N\left(1,\tfrac{y}{x}\right)$

plugging into the diff. eq. yields:

$x^{\alpha}M\left(1,\tfrac{y}{x}\right)dx + x^{\alpha}N\left(1,\tfrac{y}{x}\right)dy = 0$

$M\left(1,\tfrac{y}{x}\right)dx + N\left(1,\tfrac{y}{x}\right)dy = 0$

Let $y = ux$, then $dy = udx + xdu$

$M(1,u)\,dx + N(1,u)(udx + xdu) = 0$

$\left[M(1,u) + uN(1,u)\right]dx + xN(1,u)\,du = 0$

$$\frac{dx}{x} = \frac{-N(1,u)}{M(1,u) + uN(1,u)}du$$

$$f(x)\,dx = g(u)\,du$$

Separable

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$ydx + x(\ln x - \ln y - 1)dy = 0 \qquad y(1) = e$

Homogeneous $\Rightarrow$ let $y = ux$ or $x = vy$

$M(x,y)$ simpler than $N(x,y)$, so use

$x = vy \Rightarrow dx = vdy + ydv$

$y(vdy + ydv) + vy(\ln(vy) - \ln y - 1)dy = 0$

$y(vdy + ydv) + vy(\ln v + \cancel{\ln y - \ln y} - 1)dy = 0$

$y^2dv + \cancel{vydy} + vy\ln vdy - \cancel{vydy} = 0$

$y^2dv + vy\ln vdy = 0 \leftarrow$ Now we have a Separable D.E.

$$y^2dv = -vy\ln vdy \overset{\text{sep.}}{\Rightarrow} \frac{1}{v\ln v}dv = \frac{-1}{y}dy$$

$$\overset{\text{int.}}{\Rightarrow} \ln\left|\ln|v|\right| = -\ln|y| + C$$

$$e^{\ln\left|\ln|v|\right|} = e^{\ln|y|^{-1}+C}$$

$$\Rightarrow \ln|v| = \frac{A}{y} \qquad \left(A = e^{C}\right)$$

$$\text{sub. } v = \frac{x}{y} \Rightarrow y\ln\left|\frac{x}{y}\right| = A$$

$$y(1) = e \Rightarrow x = 1 \text{ and } y = e$$

$$e\ln\left|\frac{1}{e}\right| = A$$

$$\Rightarrow e\Big(\underbrace{\ln 1}_{0} - \underbrace{\ln e}_{1}\Big) = A \Rightarrow A = -e$$

$$\boxed{y\ln\left|\frac{x}{y}\right| = -e}$$

Let n be any real number.

The differential equation $\frac{dy}{dx}+P(x)y=f(x)y^n$

is called a **Bernoulli equation**.

The substitution $u=y^{1-n}$ reduces the equation into a linear equation.

WHY ?

$n=0$: Standard Linear Form $y'+P(x)y=f(x)$

$n=1$: still linear $y'+\left[P(x)-f(x)\right]y=0$

$n\neq 1$: $u=y^{1-n}\Rightarrow y=u^{\frac{1}{1-n}}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}$$

and $y^n=u^{\frac{n}{1-n}}$

Plug these into $\frac{dy}{dx}+P(x)y=f(x)y^n$

$$\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}+P(x)u^{\frac{1}{1-n}}=f(x)u^{\frac{n}{1-n}}$$

Multiplying by $\frac{1-n}{u^{\frac{n}{1-n}}}$ gives:

$\frac{du}{dx}+(1-n)P(x)u=(1-n)f(x)\leftarrow$ **Linear**

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$y^{1/2}\frac{dy}{dx}+y^{3/2}=1 \qquad y(0)=4$

$\frac{dy}{dx}+y=y^{-1/2}\rightarrow$ Bernoulli with $n=-1/2$

let $u=y^{1-n}\Rightarrow u=y^{3/2}$

$$y=u^{2/3}$$

$$\frac{dy}{dx}=\frac{2}{3}u^{-1/3}\frac{du}{dx} \qquad y^{-1/2}=u^{-1/3}$$

$$\frac{dy}{dx}+y=y^{-1/2}$$

$$\Downarrow$$

$\frac{2}{3}u^{-1/3}\frac{du}{dx}+u^{2/3}=u^{-1/3} \qquad \frac{du}{dx}+\frac{3}{2}u=\frac{3}{2}\leftarrow$ Now we have a D.E. in Linear form

Section 2.5 # 22

$$\frac{du}{dx}+\frac{3}{2}u=\frac{3}{2}$$

Use linear: $u'+P(x)u=Q(x)$

integrating factor $\mu=e^{\int P(x)dx}=e^{3/2x}$

Mult. both sides by μ:

$$\underbrace{e^{3/2x}\frac{du}{dx}+\frac{3}{2}e^{3/2x}u}_{(\mu u)'}=\frac{3}{2}e^{3/2x}$$

$$\left(e^{3/2x}\cdot u\right)'=\frac{3}{2}e^{3/2x}\overset{\text{int.}}{\Rightarrow}e^{3/2x}\cdot u=e^{3/2x}+C$$

$$\Rightarrow u=1+Ce^{-3/2x}$$

$$y^{3/2}=1+Ce^{-3/2x}$$

$$y(0)=4\Rightarrow 4^{3/2}-1=C$$

$$\Rightarrow y=\left(1+Ce^{-3/2x}\right)^{2/3}$$

$$C=7\quad\Rightarrow\boxed{y=\left(1+7e^{-3/2x}\right)^{2/3}}$$