Section 3.6

Cauchy-Euler

Differential Equation

$$ay'' + by' + cy = 0$$

2nd Order Homogeneous Linear Equation with Constant Coefficients

- 2nd Order → y'' is the highest derivative
- Homogeneous → set equal to zero
- Linear → y and all derivatives are raised to the first power $\left(\text{no } (y')^2\right)$
- Constant Coefficients → $a, b,$ and c are constant

Assume $y = e^{rx}$, r constant

$$cy = ce^{rx}$$
$$by' = bre^{rx}$$
$$ay'' = ar^2e^{rx}$$

$$0 = ar^2e^{rx} + bre^{rx} + ce^{rx} \quad \Rightarrow e^{rx}\left(ar^2 + br + c\right) = 0$$

e^{rx}: never zero

$ar^2 + br + c$: Auxiliary (or Indicial) Equation

The solution is based on the roots of this equation

$$ay'' + by' + cy = 0$$
$$\Downarrow$$
$$ar^2 + br + c$$

roots r_1 and r_2 based on $b^2 - 4ac$

- real distinct roots r_1 and r_2 $\left(b^2 - 4ac > 0\right)$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- repeated real roots $r_1 = r_2$ $\left(b^2 - 4ac = 0\right)$

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

- complex roots $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$ $\left(b^2 - 4ac < 0\right)$

$$y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

$$ax^2 y'' + bxy' + cy = 0$$

Cauchy-Euler Equation

2nd Order Homogeneous Linear Equation with ~~Constant~~ Variable Coefficients

with the degree on x = order of the derivative

Assume $y = x^r$, r constant

$$cy = cx^r$$
$$bxy' = bx\left(rx^{r-1}\right) \Rightarrow bxy' = brx^r$$
$$ax^2 y'' = ax^2 r(r-1)x^{r-2} \Rightarrow ax^2 y'' = ar(r-1)x^r$$
$$\Rightarrow x^r\left(ar^2 + (b-a)r + c\right) = 0$$

$x^r \neq 0$ on $(0, \infty)$

Auxiliary Equation

The solution is based on the roots of this equation

In order for a solution to exist, we need $ax^2 \neq 0$ (see Thm. 3.1) $\Rightarrow x \neq 0$ and our solution will be valid on the interval $(0, \infty)$

$$ax^2y'' + bxy' + cy = 0$$

$$\Downarrow$$

$$ar^2 + (b-a)r + c$$

roots r_1 and r_2

based on $(b-a)^2 - 4ac$

- real distinct roots r_1 and r_2

$$y = c_1x^{r_1} + c_2x^{r_2}$$

- repeated real roots $r_1 = r_2$

$$y = c_1x^{r_1} + c_2x^{r_1}\ln x$$

- complex roots $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$

$$y = x^{\alpha}\left(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\right)$$

Why?

The substitution $x = e^t$ reduces our variable coefficient equation $ax^2y'' + bxy' + cy = 0$ into a constant coefficient equation in t.

To see this, lets use $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in place of y' and y''.

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$x = e^t$$

$$\underbrace{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}}_{\text{chain rule}} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}e^t \Rightarrow \frac{dy}{dx} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)$$

Use the product rule

$$= \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^2x}{dt^2}\right)$$

(derivative of first w.r.t t) (second) (first) derivative of the second w.r.t. t

$$\frac{d^2y}{dt^2} = \left(\left(\frac{d^2y}{dx^2}\right)\cdot e^t\right)\left(e^t\right) + \left(\frac{dy}{dx}\right)\left(e^t\right)$$

$$\frac{d^2y}{dt^2} = \left(\left(\frac{d^2y}{dx^2}\right)\cdot e^t\right)\left(e^t\right) + \left(e^{-t}\frac{dy}{dt}\right)\left(e^t\right)$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = \left(\frac{d^2y}{dx^2}\right)\cdot e^{2t}$$

$$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$\Downarrow$$

$$ae^{2t}\left(e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + be^t\left(e^{-t}\frac{dy}{dt}\right) + cy = 0$$

$$\Downarrow$$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

constant coefficients

$x = e^t \quad \Rightarrow t = \ln x$

- real distinct roots r_1 and r_2

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$y = c_1 e^{r_1 \ln x} + c_2 e^{r_2 \ln x}$

$y = c_1 e^{\ln x^{r_1}} + c_2 e^{\ln x^{r_2}}$

$y = c_1 x^{r_1} + c_2 x^{r_2}$

- repeated real roots $r_1 = r_2$

$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

$y = c_1 e^{r_1 \ln x} + c_2 \ln x e^{r_1 \ln x}$

$y = c_1 e^{\ln x^{r_1}} + c_2 \ln x e^{\ln x^{r_1}}$

$y = c_1 x^{r_1} + c_2 x^{r_1} \ln x$

- complex roots $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$

$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

$y = e^{\alpha \ln x}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

$y = e^{\ln x^{\alpha}}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

$y = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

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$x^2 y'' + 3xy' - 4y = 0 \rightarrow$ Cauchy-Euler

$y = x^m \Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$x^2 y'' = m(m-1)x^{m-2}x^2 = m(m-1)x^m$

$3xy' = 3mx^{m-1}x = 3mx^m$

$-4y = -4x^m$

$0 = \underbrace{\left[m(m-1) + 3m - 4\right]}_{0} x^m \Rightarrow m^2 + 2m - 4 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2} = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

Real Distinct Roots $\Rightarrow$ $\boxed{y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}}$

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$x^3 y''' + xy' - y = 0 \rightarrow$ Cauchy-Euler

$y = x^m \Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2} \Rightarrow y''' = m(m-1)(m-2)x^{m-3}$

$x^3 y''' = m(m-1)(m-2)x^{m-3}x^3 = m(m-1)(m-2)x^m$

$xy' = mx^{m-1}x = mx^m$

$-y = -x^m$

$0 = \underbrace{\left[m(m-1)(m-2) + m - 1\right]}_{0} x^m \Rightarrow m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m-1)^3 = 0$

$m = 1,$ Repeated Root with multiplicity 3

$\Rightarrow \boxed{y = c_1 x + c_2 x \ln x + c_3 x (\ln x)^2}$