Section 3.7

Nonlinear Equations

Nonlinear 2nd Order Differential Equation has $y, y', \text{or } y''$ raised to a power.

$*$ Main problem is that the superposition principle doesn't hold

$(y'')^2 = y^2 \qquad y_1 = e^x \qquad y_1$ and y_2 are both solutions of the diff. eq.,

$\qquad y_2 = \cos x \qquad$ but $c_1 y_1 + c_2 y_2$ isn't

$*$ Can sometimes be solved by **reduction of order**.

usually when y or x are missing

$*$ Can be approximated using a **Taylor series**.

3.7 # 8

$y^2 y'' = y'$ (independent x missing)

Let $u = y' \Rightarrow u = \dfrac{dy}{dx}$

$$y'' = u' = \frac{du}{dy}\cdot\frac{dy}{dx} \qquad \Rightarrow y'' = u\frac{du}{dy}$$

The diff. eq. becomes:

$y^2 u\dfrac{du}{dy} = u$ ← Now we have a separable eq.

$du = y^{-2}dy$

$$\int du = \int y^{-2}dy \Rightarrow u = -y^{-1} + C$$

$$y' = \frac{-1}{y} + C$$

$$\frac{dy}{dx} = \frac{Cy-1}{y}$$

↑

Now we have a separable eq.

$$\frac{y}{Cy-1}dy = dx$$

$$\begin{array}{rl} & \frac{1}{C} \\ Cy-1\big) & y \\ & -\left(y-\frac{1}{C}\right) \\ \hline & \frac{1}{C} \end{array} \Rightarrow \frac{y}{Cy-1} = \frac{1}{C} + \frac{\frac{1}{C}}{Cy-1}$$

$$= \tfrac{1}{C}\left(1+\tfrac{1}{Cy-1}\right)$$

$$\tfrac{1}{C}\left(1+\tfrac{1}{Cy-1}\right)dy = dx$$

$$\boxed{= \tfrac{1}{C}\left(y+\tfrac{1}{C}\ln\left|Cy-1\right|\right) = x + K}$$

3.7 # 4

$y'' = 1 + \left(y'\right)^2$ (dependent y missing)

Let $u = y'$

$u' = y''$ (no need for chain rule)

u is a function of x directly so $u' = \dfrac{du}{dx}$

The diff. eq. becomes:

$u' = 1 + u^2$

$\dfrac{du}{dx} = 1 + u^2$ ← Now we have a separable eq.

$$\frac{du}{1+u^2} = dx$$

$$\tan^{-1} u = x + C$$

$$u = \tan\left(x+C\right)$$

$$\frac{dy}{dx} = \tan\left(x+C\right)$$

$$\int dy = \int \tan\left(x+C\right)dx$$

$$\boxed{y = -\ln\left|\cos\left(x+C\right)\right| + K}$$

changed to make the problem more interesting

3.7 # 15

$$y''=x^2+y^2-2y' \quad y(0)=-2,\ y'(0)=-1$$

Find the first 6 nonzero terms of the Taylor series solution centered at 0 (Maclaurin Series)

$$y=y(0)+y'(0)x+\tfrac{y''(0)}{2!}x^2+\tfrac{y'''(0)}{3!}x^3+\tfrac{y^{(4)}(0)}{4!}x^4+\tfrac{y^{(5)}(0)}{5!}x^5+\cdots$$

Since $y(0)=-2,\ y'(0)=-1$ we have 2 terms of the solution : $y=-2-x+\cdots$

The 3rd term involves $y''(0)$, the diff. eq. gives us y'' :

$$y''=x^2+y^2-2y' \Rightarrow y''(0)=0+\underbrace{y(0)}_{-2}{}^2-2\underbrace{y'(0)}_{-1} \quad \Rightarrow y''(0)=6$$

Take the derivative of the diff. eq. in order to get y''' (watch out for the chain rule)

$$y'''=2x+2yy'-2y'' \Rightarrow y'''(0)=0+2\underbrace{y(0)}_{-2}\underbrace{y'(0)}_{-1}-2\underbrace{y''(0)}_{6} \quad \Rightarrow y'''(0)=-8$$

Repeat the process to get $y^{(4)}$ (watch out for the product rule and chain rule)

$$y^{(4)}=2+2y'y'+2yy''-2y''' \quad \Rightarrow y^{(4)}(0)=2+2(-1)^2+2(-2)(6)-2(-8) \Rightarrow y^{(4)}(0)=-4$$

$$y^{(5)}=\underbrace{2\cdot 2y'y''+2y'y''}_{6y'y''}+2yy'''-2y^{(4)} \quad \Rightarrow y^{(5)}(0)=6(-1)(6)+2(-2)(-8)-2(-4)$$

$$\Rightarrow y^{(5)}(0)=4$$

$$\boxed{y\approx-2-x+3x^2-\tfrac{4}{3}x^3-\tfrac{1}{6}x^4+\tfrac{1}{30}x^5}$$