Section 4.1

Laplace Transforms

$t \ge 0$

$$L\left\{ f\left(t\right)\right\} = \int_0^\infty f\left(t\right)e^{-st}\,dt = F\left(s\right)$$

if the integral converges

$$L\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b$$

$$= \lim_{b\to\infty} -\underbrace{\frac{1}{se^{sb}}}_{\nearrow 0 \text{ if } s>0} - \left(-\frac{1}{s}\right)$$

$$= \frac{1}{s},\ s > 0$$

$$L\{t\}=\int_0^\infty te^{-st}\,dt \;=\lim_{b\to\infty}\left(\frac{-t}{se^{st}}-\frac{1}{s^2e^{st}}\right)\Bigg|_0^b$$

$$\begin{array}{cc} \underline{D} & \underline{I} \\ t & e^{-st} \\ & + \\ 1 & -\frac{1}{s}e^{-st} \\ & - \\ 0 & \frac{1}{s^2}e^{-st} \end{array}$$

$$=\lim_{b\to\infty}\left(\frac{-b}{se^{sb}}-\frac{1}{s^2e^{sb}}\right)-\left(0-\frac{1}{s^2}\right)$$

$$=\lim_{b\to\infty}\left(\cancel{\frac{-b}{se^{sb}}}^{\,0}-\cancel{\frac{1}{s^2e^{sb}}}^{\,0}\right)-\left(0-\frac{1}{s^2}\right)=\boxed{\frac{1}{s^2}}$$

$$s>0$$

Since $\lim_{b\to\infty}\frac{-b}{se^{sb}}\overset{L'Hop}{=}\lim_{b\to\infty}\frac{-1}{s^2e^{sb}}=0$

$$L\{t^2\}=\int_0^\infty t^2e^{-st}\,dt \;=\lim_{b\to\infty}\left(\frac{-t^2}{se^{st}}-\frac{2t}{s^2e^{st}}-\frac{2}{s^3e^{st}}\right)\Bigg|_0^b$$

$$\begin{array}{cc} \underline{D} & \underline{I} \\ t^2 & e^{-st} \\ & + \\ 2t & -\frac{1}{s}e^{-st} \\ & - \\ 2 & \frac{1}{s^2}e^{-st} \\ & + \\ 0 & -\frac{1}{s^3}e^{-st} \end{array}$$

$$=\boxed{\frac{2}{s^3}}$$

$$s>0$$

$$L\{t^3\} = \int_0^\infty t^3 e^{-st}\,dt = \lim_{b\to\infty}\left(\frac{-t^3}{se^{st}} - \frac{3t^2}{s^2e^{st}} - \frac{6t}{s^3e^{st}} - \frac{6}{s^4e^{st}}\right)_0^b$$

$\underline{D}$		$\underline{I}$
t^3	+	e^{-st}
$3t^2$	-	$-\frac{1}{s}e^{-st}$
$6t$	+	$\frac{1}{s^2}e^{-st}$
6	-	$-\frac{1}{s^3}e^{-st}$
0		$\frac{1}{s^4}e^{-st}$

$$= \boxed{\frac{6}{s^4}}$$

$$s > 0$$

$$L\{t^4\} = \int_0^\infty t^4 e^{-st}\,dt = \lim_{b\to\infty}\left(\frac{-t^4}{se^{st}} - \frac{4t^3}{s^2e^{st}} - \frac{12t^2}{s^3e^{st}} - \frac{24t}{s^4e^{st}} - \frac{24}{s^5e^{st}}\right)_0^b$$

$\underline{D}$		$\underline{I}$
t^4	+	e^{-st}
$4t^3$	-	$-\frac{1}{s}e^{-st}$
$12t^2$	+	$\frac{1}{s^2}e^{-st}$
$24t$	-	$-\frac{1}{s^3}e^{-st}$
24	+	$\frac{1}{s^4}e^{-st}$
0		$-\frac{1}{s^5}e^{-st}$

$$= \boxed{\frac{24}{s^5}}$$

$$s > 0$$

$$L\{t\}=\frac{1}{s^2}$$

$$L\{t^2\}=\frac{2}{s^3}$$

$$L\{t^3\}=\frac{6}{s^4}$$

$$L\{t^4\}=\frac{24}{s^5}$$

$$\vdots \quad \vdots \quad \vdots$$

$$L\{t^n\}=\frac{n!}{s^{n+1}}$$ for integer $n>0$, $s>0$

$$L\{e^{at}\}=\int_0^\infty e^{at}e^{-st}\,dt=\int_0^\infty e^{(1-s)t}\,dt \quad =\int_0^\infty e^{-(s-a)t}\,dt$$

$$=\lim_{b\to\infty}\left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^b$$

$$=\lim_{b\to\infty}-\underbrace{\frac{1}{(s-a)e^{(s-a)b}}}_{\nearrow 0 \text{ if } s-a>0 \Rightarrow s>a}-\left(-\frac{1}{(s-a)}\right)$$

$$=\frac{1}{(s-a)},\, s>a$$

$$L\{\sin(at)\}=\int_0^\infty e^{-st}\sin(at)\,dt \quad =\lim_{b\to\infty}-\left.\frac{\sin(at)}{se^{st}}\right|_0^b+\frac{a}{s}\int_0^b e^{-st}\cos(at)\,dt$$

$u=\sin(at) \quad dv=e^{-st}dt$

$du=a\cos(at) \quad v=-\frac{1}{s}e^{-st}$

$$\Rightarrow\int_0^\infty e^{-st}\sin(at)\,dt=\lim_{b\to\infty}0+\frac{a}{s}\left\{\int_0^b e^{-st}\cos(at)\,dt\right\}$$

$u=\cos(at) \quad dv=e^{-st}dt$

$du=-a\sin(at) \quad v=-\frac{1}{s}e^{-st}$

$$\Rightarrow\int_0^\infty e^{-st}\sin(at)\,dt=\frac{a}{s}\left\{\lim_{b\to\infty}-\left.\frac{\cos(at)}{se^{st}}\right|_0^b-\frac{a}{s}\int_0^b e^{-st}\sin(at)\,dt\right\}$$

$$\Rightarrow\int_0^\infty e^{-st}\sin(at)\,dt=\lim_{b\to\infty}\frac{a}{s}\left\{\frac{1}{s}-\frac{a}{s}\int_0^b e^{-st}\sin(at)\,dt\right\}$$

$$\Rightarrow\underbrace{\int_0^\infty e^{-st}\sin(at)\,dt}_{I}=\frac{a}{s^2}-\frac{a^2}{s^2}\underbrace{\int_0^\infty e^{-st}\sin(at)\,dt}_{I}$$

$$\Rightarrow I\left(1+\frac{a^2}{s^2}\right)=\frac{a}{s^2} \quad \Rightarrow I\left(\frac{s^2+a^2}{s^2}\right)\frac{s^2}{s^2+a^2}=\frac{a}{s^2}\cdot\frac{s^2}{s^2+a^2} \quad \Rightarrow I=\int_0^\infty e^{-st}\sin(at)\,dt=\boxed{\frac{a}{s^2+a^2}}$$

$s>0$

$$L\{\sinh(at)\}=\int_0^\infty \sinh(at)e^{-st}dt \quad =\int_0^\infty\left(\frac{e^{at}-e^{-at}}{2}\right)e^{-st}dt \quad =\frac{1}{2}\int_0^\infty e^{(a-s)t}-e^{-(a+s)t}dt$$

$$=\lim_{b\to\infty}\frac{1}{2}\left.\left(\frac{1}{a-s}e^{(a-s)t}+\frac{1}{a+s}e^{-(a+s)t}\right)\right|_0^b, \text{converges if } \underbrace{a-s<0}_{s>a} \ \& \ \underbrace{a+s>0}_{s>-a}\Rightarrow s>|a|$$

$$=\lim_{b\to\infty}\frac{1}{2}\left.\left(\frac{1}{(a-s)e^{-(a-s)t}}+\frac{1}{(a+s)e^{(a+s)t}}\right)\right|_0^b \quad =\frac{1}{2}\left(\frac{1}{(a-s)}+\frac{1}{(a+s)}\right)$$

$$=\frac{1}{2}\left(\frac{a+s}{a^2-s^2}+\frac{a-s}{a^2-s^2}\right)$$

$$=\frac{1}{2}\left(\frac{2a}{a^2-s^2}\right)$$

$$=\boxed{\frac{a}{a^2-s^2}}$$

$s>|a|$

$$L\left\{t^n e^{at}\right\} = \int_0^\infty t^n e^{at} e^{-st}\,dt = \int_0^\infty t^n e^{(a-s)t}\,dt$$

$$= \int_0^\infty t^n e^{-(s-a)t}\,dt,\ s > a$$

$\underline{D}$		$\underline{I}$
t^n	+	$e^{-(s-a)t}$
nt^{n-1}	-	$-\frac{1}{s-a}e^{-(s-a)t}$
$n(n-1)t^{n-2}$	+	$\frac{1}{(s-a)^2}e^{-(s-a)t}$
$n(n-1)(n-2)t^{n-3}$		$-\frac{1}{(s-a)^3}e^{-(s-a)t}$
$\vdots$		$\vdots$
$\vdots$		$\vdots$
$\vdots$		$\vdots$
0		$\frac{(-1)^n}{(s-a)^{n+1}}e^{-(s-a)t}$

$$= \lim_{b\to\infty}\left(\frac{-t^n}{(s-a)e^{(s-a)t}} - \frac{nt^{n-1}}{(s-a)^2 e^{(s-a)t}} - \cdots - \frac{n!t}{(s-a)^n e^{(s-a)t}} - \frac{n!}{(s-a)^{n+1} e^{(s-a)t}}\right)_0^b$$

$$= \boxed{\frac{n!}{(s-a)^{n+1}}}$$

for integer $n > 0$

$s > a$

The Laplace transform is linear:

$$L\{cf(t) + kg(t)\} = cL\{f(t)\} + kL\{g(t)\}$$

(since the integral is linear)

$$L\left\{5t^2 - 7e^{6t} + 3\sin 4t\right\} = 5L\left\{t^2\right\} - 7L\left\{e^{6t}\right\} + 3L\{\sin 4t\}$$

$$= 5\left(\frac{2}{s^3}\right) - 7\left(\frac{1}{s-6}\right) + 3\left(\frac{4}{s^2+16}\right)$$

$$L\left\{5t^2 - 7e^{6t} + 3\sin 4t\right\} = \frac{10}{s^3} - \frac{7}{s-6} + \frac{12}{s^2+16}$$