Section 4.2
Inverse Laplace Transform and Transforms of Derivatives

Section 4.2 - Rimmer

	Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$	
1	1	$\frac{1}{s}$ $\quad s > 0$	✓
2	t (unit-ramp function)	$\frac{1}{s^2}$ $\quad s > 0$	✓
3	t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$ $\quad s > 0$	✓
4	e^{at}	$\frac{1}{s-a}$ $\quad s > a$	✓
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$ $\quad s > 0$	✓
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$ $\quad s > 0$	
7	$t^n g(t)$, for $n = 1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$	
8	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$ $\quad s > \lvert\omega\rvert$	
9	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ $\quad s > \lvert\omega\rvert$	
10	$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$ Scale property	
11	$e^{at}g(t)$	$G(s-a)$ Shift property	
12	$e^{at}t^n$, for $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$ $\quad s > a$	✓

http://www.intmath.com/Laplace/2_lap_defn.php

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$$\frac{1}{s^2+s-20}=\frac{1}{(s+5)(s-4)}=\frac{A}{s+5}+\frac{B}{s-4}$$

cover up method:

let $s=-5$: $\frac{1}{\cancel{(s+5)}(s-4)}=\frac{1}{(-5-4)}\Rightarrow A=\frac{-1}{9}$

let $s=4$: $\frac{1}{(s+5)\cancel{(s-4)}}=\frac{1}{(4+5)}\Rightarrow B=\frac{1}{9}$

$$\frac{1}{(s+5)(s-4)}=\frac{-1/9}{s+5}+\frac{1/9}{s-4}$$

$$L^{-1}\left\{\frac{1}{s^2+s-20}\right\}=L^{-1}\left\{\frac{-1/9}{s+5}+\frac{1/9}{s-4}\right\}=\frac{-1}{9}L^{-1}\left\{\frac{1}{s+5}\right\}+\frac{1}{9}L^{-1}\left\{\frac{1}{s-4}\right\}=\boxed{\frac{-1}{9}e^{-5t}+\frac{1}{9}e^{4t}}$$

$$L\{f'(t)\}=\int_0^\infty f'(t)e^{-st}\,dt \qquad \begin{matrix} u=e^{-st} & dv=f'(t)\,dt \\ du=-se^{-st} & v=f(t) \end{matrix}$$

$$=\lim_{b\to\infty}\left.\frac{f(t)}{e^{st}}\right|_0^b+s\int_0^b f(t)e^{-st}dt$$

$$=\lim_{b\to\infty}\left.\frac{f(b)}{e^{sb}}-\frac{f(0)}{1}\right|_0^b+sL\{f(t)\}$$

$$\Rightarrow \boxed{L\{f'(t)\}=sL\{f(t)\}-f(0)}$$

$$L\{f''(t)\} = \int_0^\infty f''(t)e^{-st}\,dt \qquad \begin{matrix} u = e^{-st} & dv = f''(t)\,dt \\ du = -se^{-st} & v = f'(t) \end{matrix}$$

$$= \lim_{b\to\infty} \left.\frac{f'(t)}{e^{st}}\right|_0^b + s\int_0^b f'(t)e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left.\frac{f'(b)}{e^{sb}} - \frac{f'(0)}{1}\right|_0^b + sL\{f'(t)\}$$

$$= -f'(0) + sL\{f'(t)\}$$

$$= -f'(0) + s\left(sL\{f(t)\} - f(0)\right)$$

$$\Rightarrow \boxed{L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)}$$

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$y'' + 9y = e^t \qquad y(0) = 0, y'(0) = 0$

$L\{y''\} + 9L\{y\} = L\{e^t\}$

$$s^2L\{y\} - s\cdot\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 9L\{y\} = \frac{1}{s-1}$$

$$L\{y\} = \frac{1}{(s-1)(s^2+9)} \Rightarrow y(t) = L^{-1}\left\{\frac{1}{(s-1)(s^2+9)}\right\}$$

$$\frac{1}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$$

$$1 = A(s^2+9) + \underbrace{(Bs+C)(s-1)}_{Bs^2-Bs+Cs-C}$$

$$1 = \underbrace{(A+B)}_{0}s^2 + \underbrace{(C-B)}_{0}s + \underbrace{(9A-C)}_{1}$$

$\Rightarrow A = -B, B = C$, so $A = -C \Rightarrow -10C = 1$

$C = -1/10, B = -1/10$, and $A = 1/10$

$$y(t) = L^{-1}\left\{\frac{1}{(s-1)(s^2+9)}\right\} = L^{-1}\left\{\frac{1/10}{(s-1)} + \frac{-1/10(s+1)}{(s^2+9)}\right\}$$

$$y(t) = \frac{1}{10}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{10}L^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{1}{10}L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2+9}\right\} = L^{-1}\left\{\frac{3}{3}\cdot\frac{1}{s^2+9}\right\} = \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = \frac{1}{10}e^t - \frac{1}{10}\cos(3t) - \frac{1}{30}\sin(3t)}$$