Section 4.4
More Transform Properties

- Derivatives of Transforms
- Transform of a Convolution
- Transform of a periodic Function

Laplace Transform $L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$

$$\frac{d}{ds}(L\{f(t)\}) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}\left(f(t)e^{-st}\right)dt = -\int_0^\infty tf(t)e^{-st}dt$$

$$L\{tf(t)\} = -\frac{d}{ds}(L\{f(t)\})$$

$$\frac{d^2}{ds^2}(L\{f(t)\}) = \frac{d}{ds}\left[-\int_0^\infty tf(t)e^{-st}dt\right] = -\int_0^\infty \frac{\partial}{\partial s}\left(tf(t)e^{-st}\right)dt = \int_0^\infty t^2 f(t)e^{-st}dt$$

$$L\{t^2 f(t)\} = \frac{d^2}{ds^2}(L\{f(t)\})$$

$$\vdots \quad \vdots$$

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(L\{f(t)\})}$$

We derived

$$L\{\sinh(\omega t)\} = \frac{\omega}{s^2-\omega^2}$$

$$\frac{d}{ds}\left(\frac{\omega}{s^2-\omega^2}\right) = \frac{d}{ds}\left[\omega\left(s^2-\omega^2\right)^{-1}\right]$$

$$= -\omega\left(s^2-\omega^2\right)^{-2}(2s)$$

$$= \frac{-2\omega s}{\left(s^2-\omega^2\right)^2}$$

$$-\frac{d}{ds}\left(L\{f(t)\}\right) = L\{tf(t)\}$$

$$L\{t\sinh(\omega t)\} = \frac{2\omega s}{\left(s^2-\omega^2\right)^2}$$

Let f and g be piecewise continuous functions on $[0,\infty)$.

$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$ is called the **convolution** of f and g.

Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous functions on $[0,\infty)$ and of exponential order, then

$$L\{f * g\} = L\{f(t)\}\cdot L\{g(t)\}$$

If $f(t)$ is a piecewise continuous function on $[0,\infty)$, of exponential order, and periodic with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$