Section 5.1 Series Solutions to Linear Diff. Eq. about Ordinary Pts.

Linear Second-Order Differential Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Divide by $a_2(x)$ to put the above equation in standard form.

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

$x = x_0$ is an **ordinary point** if $a_2(x) \neq 0$.

$$y'' + P(x)y' + Q(x)y = 0$$

$x = x_0$ is an **singular point** if $a_2(x) = 0$.

In this section (5.1) we study power series solutions about ordinary points.

Power Series centered about $x = a$.

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

This defines a function $f(x)$ whose domain is the interval of convergence.
If the radius of convergence $R > 0$, then $f(x)$ is continuous, differentiable, and integrable on $(a-R, a+R)$

For most problems in this section we will have $a = 0$.

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

y' and y'' are found by term - by term differentiation.

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots$$

Theses series get substituted into the differential equation in place of y, y' and y''.

The solution y is then found by shifting indices and finding a recurrence relation that determine the coefficients.

We actually find two linearly independent solutions y_1 and y_2 and the solution y is a linear combination of these.

$$y'' - 2xy' + y = 0$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$-2xy'(x) = -2x \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} -2n c_n x^n$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$0 = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} -2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

Our goal is to :

a) get every summation to have the same power on x (shift or rename indices),

b) peel off terms so that all summations start at the same value,

c) combine the summations into one (from this we find the recurrence relation)

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$$0=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty}-2nc_n x^n+\sum_{n=0}^{\infty}c_n x^n$$

$\Downarrow$ Let $k=n-2$ $\Rightarrow n-1=k+1$ $\Rightarrow n=k+2$ When $n=2, k=0$ — index **shift**

$\Downarrow$ Let $k=n$ — index **renaming**

$\Downarrow$ Let $k=n$ — index **renaming**

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=1}^{\infty}-2kc_k x^k+\sum_{k=0}^{\infty}c_k x^k=0$$

"**peel off**" 1 term: $\underline{k=0}$ $2c_2$

"**peel off**" 0 terms

"**peel off**" 1 term: $\underline{k=0}$ c_0

$$2c_2+\sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=1}^{\infty}-2kc_k x^k+c_0+\sum_{k=1}^{\infty}c_k x^k=0$$

$$2c_2+c_0+\sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2}-2kc_k+c_k\right]x^k=0$$

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$$2c_2+c_0+\sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2}-2kc_k+c_k\right]x^k=0$$

$\Downarrow$ $2c_2+c_0=0$

$\Downarrow$ $\left[(k+2)(k+1)c_{k+2}+(-2k+1)c_k\right]=0$

$\boxed{c_2=-\tfrac{1}{2}c_0}$

$$c_{k+2}=\frac{(2k-1)}{(k+2)(k+1)}c_k \qquad k=1,2,3,\ldots$$

recurrence relation

$k=1\Rightarrow$ $c_3=\frac{1}{(3)(2)}c_1$ $\boxed{c_3=\tfrac{1}{6}c_1}$

$k=2\Rightarrow$ $c_4=\frac{3}{(4)(3)}c_2$ $\quad c_4=\tfrac{1}{4}c_2 \quad c_4=\tfrac{1}{4}\left(\tfrac{-1}{2}c_0\right)$ $\boxed{c_4=\tfrac{-1}{8}c_0}$

$k=3\Rightarrow$ $c_5=\frac{5}{(5)(4)}c_3$ $\quad c_5=\tfrac{1}{4}c_3 \quad c_5=\tfrac{1}{4}\left(\tfrac{1}{6}c_1\right)$ $\boxed{c_5=\tfrac{1}{24}c_1}$

$k=4\Rightarrow$ $c_6=\frac{7}{(6)(5)}c_4$ $\quad c_6=\tfrac{7}{30}c_4 \quad c_6=\tfrac{7}{30}\left(\tfrac{-1}{8}c_0\right)$ $\boxed{c_6=\tfrac{-7}{240}c_0}$

$k=5\Rightarrow$ $c_7=\frac{9}{(7)(6)}c_5$ $\quad c_7=\tfrac{3}{14}c_5 \quad c_7=\tfrac{3}{14}\left(\tfrac{1}{24}c_1\right)$ $\boxed{c_7=\tfrac{1}{112}c_1}$

$$y(x)=\sum_{n=0}^{\infty}c_nx^n = c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+\cdots$$

$$=c_0+c_1x+\left(\tfrac{-1}{2}c_0\right)x^2+\left(\tfrac{1}{6}c_1\right)x^3+\left(\tfrac{-1}{8}c_0\right)x^4+\left(\tfrac{1}{24}c_1\right)x^5+\left(\tfrac{-7}{240}c_0\right)x^6+\left(\tfrac{1}{112}c_1\right)x^7+\cdots$$

$$\boxed{\begin{aligned} y(x)=c_0\left(1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{7}{240}x^6-\cdots\right)\\ +c_1\left(x+\tfrac{1}{6}x^3+\tfrac{1}{24}x^5+\tfrac{1}{112}x^7+\cdots\right)\end{aligned}}$$

If given initial conditions, we can find c_0 and c_1

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