Section 5.2
Solutions to Linear Diff. Eq. about Singular Pts.

Linear Second-Order Differential Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Divide by $a_2(x)$ to put the above equation in standard form.

$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$ $x = x_0$ is an **ordinary point** if $a_2(x) \neq 0.$

$y'' + P(x)y' + Q(x)y = 0$ $x = x_0$ is an **singular point** if $a_2(x) = 0.$

In this section (5.2) we study power series solutions about singular points.

There are 2 types of singular points:

$x = x_0$ is a **regular singular point** if both $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are analytic functions at x_0.

has a Taylor series expansion about x_0

$x = x_0$ is an **irregular singular point** if it is a singular point that is not regular

Visual Check:

If $a_2(x), a_1(x),$ and $a_0(x)$ are polynomials, $x = x_0$ is a **regular singular point** if both $x - x_0$ appears at most to the **first power** in $P(x)$ and at most to the **second power** in $Q(x)$.

$2x^2 y'' - xy' + (x^2+1)y = 0$ $x = 0$ is a singular point

$$y'' - \frac{x}{2x^2} y' + \frac{(x^2+1)}{2x^2} y = 0$$

$$y'' - \frac{1}{2x} y' + \frac{x^2+1}{2x^2} y = 0$$

Standard Form $P(x) = -\frac{1}{2x}$ and $Q(x) = \frac{x^2+1}{2x^2}$

Visual Check:

$x = 0$ is a regular singular point since x appears at most to the first power in $P(x)$ and at most to the second power in $Q(x)$.

From the definition:

$p(x) = xP(x) = -\frac{1}{2}$ and $q(x) = x^2 Q(x) = \frac{1}{2}x^2 + \frac{1}{2}$

$p(x)$ and $q(x)$ are both analytic functions so

$x = 0$ is a regular singular point

Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential eq.

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$

then there exists at least one solution of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is a constant to be determined by the indicial equation.

The series will converge at least on some interval $0 < x - x_0 < R$.

Let $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r} + \cdots = x^r \left(c_0 + c_1 x + c_2 x^2 + \cdots\right)$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} = rc_0 x^{r-1} + (1+r) c_1 \underbrace{x^{1+r-1}}_{x^r} + (2+r) c_2 \underbrace{x^{2+r-1}}_{x^{r+1}} + \cdots$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$2x^2y'' - xy' + (x^2+1)y = 0 \qquad 2x^2y'' - xy' + x^2y + y = 0$

$$2x^2y'' = 2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2} = \sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r}$$

$$-xy' = -x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} = -\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}$$

$$x^2y = x^2\sum_{n=0}^{\infty}c_nx^{n+r} = \sum_{n=0}^{\infty}c_nx^{n+r+2}$$

$$y = \sum_{n=0}^{\infty}c_nx^{n+r}$$

$$0 = \sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r+2} + \sum_{n=0}^{\infty}c_nx^{n+r}$$

Our goal is to :

a) get every summation to have the same power on x (but first factor out x^r),

b) peel off terms so that all summations start at the same value,

c) combine the summations into one (this gives the recurrence relation)

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$$0 = \sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r+2} + \sum_{n=0}^{\infty}c_nx^{n+r}$$

$$0 = x^r\left[\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_nx^{n} - \sum_{n=0}^{\infty}(n+r)c_nx^{n} + \sum_{n=0}^{\infty}c_nx^{n+2} + \sum_{n=0}^{\infty}c_nx^{n}\right]$$

$\Downarrow$ Let $k = n$

$\Downarrow$ Let $k = n$

$\Downarrow$ Let $k = n+2$

$\Rightarrow n = k-2$

When $n = 0, k = 2$

$\Downarrow$ Let $k = n$

$$0 = x^r\left[\sum_{k=0}^{\infty}\left[2(k+r)(k+r-1)-(k+r)+1\right]c_kx^k + \sum_{k=2}^{\infty}c_{k-2}x^k\right]$$

“peel off”
2 terms

$$0 = x^r\left[(2r(r-1)-r+1)c_0 + (2(r+1)r-(r+1)+1)c_1x + \sum_{k=2}^{\infty}\left[(2(k+r)(k+r-1)-(k+r)+1)c_k + c_{k-2}\right]x^k\right]$$

$(2r(r-1)-r+1)c_0 \quad \Rightarrow (2r^2-3r+1)c_0 = 0 \quad \Rightarrow (2r-1)(r-1) = 0$

indicial equation

$\Rightarrow r_1 = \frac{1}{2}$ and $r_2 = 1$

$(2(r+1)r-(r+1)+1)c_1x \quad \Rightarrow (2r^2+r)c_1x$

for r_1 and $r_2, 2r^2+r \neq 0$

$\Rightarrow c_1 = 0$

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$$(2(k+r)(k+r-1)-(k+r)+1)c_k + c_{k-2} = 0 \quad k = 2,3,4,\ldots$$

For $r_1 = \frac{1}{2}$

$$(2(k+\tfrac{1}{2})(k-\tfrac{1}{2})-(k+\tfrac{1}{2})+1)c_k + c_{k-2} = 0$$

$$(2(k^2-\tfrac{1}{4})-k+\tfrac{1}{2})c_k + c_{k-2} = 0$$

$$(2k^2-k)c_k + c_{k-2} = 0$$

$$\underbrace{c_k = -\frac{c_{k-2}}{k(2k-1)}}_{\textbf{recurrence relation}} \quad k = 2,3,4,\ldots$$

$k=2:\ c_2 = -\frac{c_0}{2(3)}$ $\qquad c_2 = -\frac{1}{6}c_0$ $\qquad$ Let $c_0 = 1$

$k=3:\ c_3 = -\frac{c_1}{3(5)}$ $\qquad c_3 = 0$

$k=4:\ c_4 = -\frac{c_2}{4(7)}$ $\qquad c_4 = (\frac{-1}{28})(-\frac{1}{6})c_0$

$c_4 = \frac{1}{168}c_0$

$$\boxed{y_1(x) = x^{\frac{1}{2}}\left(1 - \tfrac{1}{6}x^2 + \tfrac{1}{168}x^4 - \cdots\right)}$$

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$$(2(k+r)(k+r-1)-(k+r)+1)c_k + c_{k-2} = 0 \quad k = 2,3,4,\ldots$$

For $r_2 = 1$

$$(2k(k+1)-(k+1)+1)c_k + c_{k-2} = 0$$

$$(2k^2+k)c_k + c_{k-2} = 0$$

$$\underbrace{c_k = -\frac{c_{k-2}}{k(2k+1)}}_{\textbf{recurrence relation}} \quad k = 2,3,4,\ldots$$

$k=2:\ c_2 = -\frac{c_0}{2(5)}$ $\qquad c_2 = -\frac{1}{10}c_0$ $\qquad$ Let $c_0 = 1$

$k=3:\ c_3 = -\frac{c_1}{3(7)}$ $\qquad c_3 = 0$

$k=4:\ c_4 = -\frac{c_2}{4(9)}$ $\qquad c_4 = (\frac{-1}{36})(-\frac{1}{10})c_0$

$c_4 = \frac{1}{360}c_0$

$$\boxed{y_2(x) = x\left(1 - \tfrac{1}{10}x^2 + \tfrac{1}{360}x^4 - \cdots\right)}$$

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$$2x^2y'' - xy' + \left(x^2+1\right)y = 0$$

Solution

$y(x) = C_1y_1 + C_2y_2$

$$\boxed{y(x) = C_1x^{\frac{1}{2}}\left(1 - \tfrac{1}{6}x^2 + \tfrac{1}{168}x^4 - \cdots\right) + C_2x\left(1 - \tfrac{1}{10}x^2 + \tfrac{1}{360}x^4 - \cdots\right)}$$

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You can find the indicial equation at the beginning of the problem.

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ Divide by $a_2(x)$ to put the above equation in standard form.

$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$

$y'' + P(x)y' + Q(x)y = 0$ **Standard Form**

Assume we have a regular singular point at $x_0 = 0$.

Find $p(x) = xP(x)$ and $q(x) = x^2Q(x)$

Let

$p(x) = xP(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $q(x) = x^2Q(x) = b_0 + b_1x + b_2x^2 + \cdots$

The indicial equation is then : $r(r-1) + a_0r + b_0 = 0$

$2x^2y'' - xy' + \left(x^2+1\right)y = 0$

$y'' - \frac{x}{2x^2}y' + \frac{\left(x^2+1\right)}{2x^2}y = 0$

$y'' - \frac{1}{2x}y' + \frac{x^2+1}{2x^2}y = 0$

$P(x) = -\frac{1}{2x}$ and $Q(x) = \frac{x^2+1}{2x^2}$

$p(x) = xP(x) = -\frac{1}{2}$

$q(x) = x^2Q(x) = \frac{1}{2}x^2 + \frac{1}{2}$

$a_0 = \frac{-1}{2}, b_0 = \frac{1}{2}$

$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$

$r^2 - \frac{3}{2}r + \frac{1}{2} = 0$

$\boxed{2r^2 - 3r + 1 = 0}$

indicial equation