8.10 Orthogonal Matrices

Preliminary Results:

complex number	conjugate	magnitude
$\mathbf{z} = a + bi \quad (i^2 = -1)$	$\bar{\mathbf{z}} = a - bi$	$\|\mathbf{z}\| = \sqrt{a^2 + b^2}$

$$\mathbf{z}\bar{\mathbf{z}} = (a+bi)(a-bi) = (a^2+b^2) + 0i = a^2 + b^2 \quad \Rightarrow \mathbf{z}\bar{\mathbf{z}} = \|\mathbf{z}\|^2$$

$\mathbf{z} = \bar{\mathbf{z}} \Rightarrow b = 0 \Rightarrow \mathbf{z}$ is a real number

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$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

x is $n \times 1$

$x^T = (x_1, x_2, \dots, x_n)$

x^T is $1 \times n$

$x^T x$ has size $(1 \times \cancel{n})(\cancel{n} \times 1) = 1 \times 1$ (a scalar)

$$x^T x = x_1^2 + x_2^2 + \cdots x_n^2 = x \cdot x \quad (\cdot = \text{dot product})$$

$$x^T x = x \cdot x = \|x\|^2$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

y is $n \times 1$

$$x \cdot y = x^T y = y^T x$$

$$x \cdot y = 0 \Rightarrow x \perp y$$

Let A be $n \times n$, if $A = A^T$, then A is **symmetric**.

Theorem :

Let A be a symmetric matrix with real entries, then the eigenvalues of A are real.

Proof :

Suppose $A\mathbf{x} = \lambda \mathbf{x}$ $\begin{cases} \text{where } \mathbf{x} \text{ is a nonzero eigenvector} \\ \text{with eigenvalue } \lambda \end{cases}$

Taking the conjugate yields $\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$

A is real so $A = \overline{A} \Rightarrow A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$

Taking the transpose yields $\overline{\mathbf{x}}^T A^T = \overline{\lambda}\overline{\mathbf{x}}^T$

Let $\mathbf{x}^* = \overline{\mathbf{x}}^T$ and A is symmetric $\Rightarrow \mathbf{x}^* A = \overline{\lambda}\mathbf{x}^*$

Right mult. by $\mathbf{x}$ yields $\mathbf{x}^* A\mathbf{x} = \overline{\lambda}\mathbf{x}^*\mathbf{x}$

$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{x}^*\lambda\mathbf{x} = \overline{\lambda}\mathbf{x}^*\mathbf{x}$

$\Rightarrow 0 = \overline{\lambda}\mathbf{x}^*\mathbf{x} - \lambda\mathbf{x}^*\mathbf{x} \Rightarrow (\overline{\lambda} - \lambda)\mathbf{x}^*\mathbf{x} = 0$

$\underbrace{\mathbf{x}^*\mathbf{x} \neq \mathbf{0}}_{\text{see note}} \Rightarrow \overline{\lambda} = \lambda \Rightarrow \boxed{\lambda \text{ is real}}$

Note:

Let $\mathbf{x} = \begin{pmatrix} \mathbf{z_1} \\ \mathbf{z_2} \\ \vdots \\ \mathbf{z_n} \end{pmatrix}$, then $\mathbf{x}^* = (\overline{\mathbf{z_1}} \quad \overline{\mathbf{z_2}} \quad \cdots \quad \overline{\mathbf{z_n}})$

$\mathbf{x}^*\mathbf{x} = \overline{\mathbf{z_1}}\mathbf{z_1} + \overline{\mathbf{z_2}}\mathbf{z_2} + \cdots + \overline{\mathbf{z_n}}\mathbf{z_n} =$

$\mathbf{x}^*\mathbf{x} = \|\mathbf{z_1}\|^2 + \|\mathbf{z_2}\|^2 + \cdots + \|\mathbf{z_n}\|^2$

$\mathbf{x}^*\mathbf{x} = 0 \Rightarrow \mathbf{z_1} = \mathbf{z_2} = \cdots = \mathbf{z_n} = 0 \Rightarrow \mathbf{x} = \mathbf{0}$

$\mathbf{x} \neq \mathbf{0} \Rightarrow \mathbf{x}^*\mathbf{x} \neq 0$

Theorem : 8.10

Let A be a symmetric matrix, then eigenvectors corresponding to different eigenvalues are orthogonal.

Proof :

Suppose $A\mathbf{x_1} = \lambda_1\mathbf{x_1}$ and $A\mathbf{x_2} = \lambda_2\mathbf{x_2}$ {where $\lambda_1 \neq \lambda_2, \mathbf{x}_1, \mathbf{x}_2$ are nonzero eigenvectors

Taking the transpose yields : $(A\mathbf{x_1})^T = (\lambda_1\mathbf{x_1})^T \Rightarrow \mathbf{x_1}^T A^T = \lambda_1\mathbf{x_1}^T$

A is symmetric $\Rightarrow \mathbf{x_1}^T A = \lambda_1\mathbf{x_1}^T$

Right mult. by $\mathbf{x_2}$ yields $\mathbf{x_1}^T A\mathbf{x_2} = \lambda_1\mathbf{x_1}^T\mathbf{x_2}$

$A\mathbf{x_2} = \lambda_2\mathbf{x_2} \Rightarrow \mathbf{x_1}^T\lambda_2\mathbf{x_2} = \lambda_1\mathbf{x_1}^T\mathbf{x_2}$

$\Rightarrow 0 = \lambda_1\mathbf{x_1}^T\mathbf{x_2} - \mathbf{x_1}^T\lambda_2\mathbf{x_2} \Rightarrow (\lambda_1 - \lambda_2)\mathbf{x_1}^T\mathbf{x_2} = 0$

$(\lambda_1 - \lambda_2) \neq 0 \Rightarrow \mathbf{x_1}^T\mathbf{x_2} = 0$

$\Rightarrow \mathbf{x_1} \cdot \mathbf{x_2} = 0$

$\Rightarrow \boxed{\mathbf{x_1} \perp \mathbf{x_2}}$

In addition, for an $n \times n$ symmetric matrix A, there will be n linearly independent eigenvectors.

Let A be $n \times n$, if $A^T = A^{-1}$, then A is **orthogonal**.

Theorem :

A is an orthogonal matrix if and only if it's columns form an orthonormal set.

Proof : 3×3 case

Let the columns of A be $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$.

orthogonal unit vectors

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \mathbf{X}_3^T \end{pmatrix}$$

$\uparrow \; \uparrow \; \uparrow$

$\mathbf{X_1}\,\mathbf{X_2}\,\mathbf{X_3}$

A is orthogonal $\Rightarrow A^T A = I$

$$A^T A = \begin{pmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \mathbf{X}_3^T \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1^T\mathbf{X}_1 & \mathbf{X}_1^T\mathbf{X}_2 & \mathbf{X}_1^T\mathbf{X}_3 \\ \mathbf{X}_2^T\mathbf{X}_1 & \mathbf{X}_2^T\mathbf{X}_2 & \mathbf{X}_2^T\mathbf{X}_3 \\ \mathbf{X}_3^T\mathbf{X}_1 & \mathbf{X}_3^T\mathbf{X}_2 & \mathbf{X}_3^T\mathbf{X}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1\cdot\mathbf{X}_1 & \mathbf{X}_1\cdot\mathbf{X}_2 & \mathbf{X}_1\cdot\mathbf{X}_3 \\ \mathbf{X}_2\cdot\mathbf{X}_1 & \mathbf{X}_2\cdot\mathbf{X}_2 & \mathbf{X}_2\cdot\mathbf{X}_3 \\ \mathbf{X}_3\cdot\mathbf{X}_1 & \mathbf{X}_3\cdot\mathbf{X}_2 & \mathbf{X}_3\cdot\mathbf{X}_3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \mathbf{X}_1\cdot\mathbf{X}_1 = 1 \\ \mathbf{X}_2\cdot\mathbf{X}_2 = 1 \\ \mathbf{X}_3\cdot\mathbf{X}_3 = 1 \end{matrix} \text{ and } \begin{matrix} \mathbf{X}_1\cdot\mathbf{X}_2 = 0 \\ \mathbf{X}_1\cdot\mathbf{X}_3 = 0 \\ \mathbf{X}_2\cdot\mathbf{X}_3 = 0 \end{matrix}$$

unit vectors orthogonal

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To construct an $n \times n$ orthogonal matrix we need n linearly independent, orthogonal unit vectors.

These become the columns of the orthogonal matrix.

$S = \left\{ \langle 0, \frac{-3}{5}, \frac{4}{5} \rangle, \langle 0, \frac{4}{5}, \frac{3}{5} \rangle, \langle 1, 0, 0 \rangle \right\}$

- each vector is a unit vector
- they are orthogonal

Are the vectors linearly independent?

Make the vectors rows of a matrix
find the rank of this matrix
if the rank = the # of vectors,
then the vectors are linearly independent

$$M = \begin{pmatrix} 0 & \frac{-3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \end{pmatrix} \underset{\Rightarrow}{\text{row operations}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

$rank(M) = 3 \Rightarrow$ the vectors are linearly independent

$\Rightarrow$ the matrix with these vectors as columns is orthogonal

$$A = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix} \text{ is orthogonal.}$$

It would be ideal to have a way to guarantee n linearly independent orthogonal unit vectors.

If A is a symmetric matrix, then we can construct an orthogonal matrix using the eigenvectors of A.

We know that for an $n \times n$ symmetric matrix A, there will be n linearly independent eigenvectors.

We know that when A is symmetric, the eigenvectors corresponding to different eigenvalues are orthogonal.

When A is symmetric, a repeated eigenvalue with multiplicity m yields m linearly independent eigenvectors that can be chosen to be orthogonal.

Putting these facts together implies that

An $n \times n$ symmetric matrix has n linearly independent orthogonal eigenvectors.

Making the vectors unit vectors can be done by scaling each by the reciprocal of its magnitude.

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$$A=\begin{pmatrix}-8&5&4\\5&3&1\\4&1&0\end{pmatrix}$$

$\lambda_1=0 \quad \lambda_2=6 \quad \lambda_3=-11$

$$v_1=\begin{pmatrix}1\\-4\\7\end{pmatrix}\quad v_2=\begin{pmatrix}1\\2\\1\end{pmatrix}\quad v_3=\begin{pmatrix}-3\\1\\1\end{pmatrix}$$

Symmetric matrix with distinct eigenvalues

$\Downarrow$

linearly independent orthogonal eigenvectors

$\|v_1\|=\sqrt{66} \quad \|v_2\|=\sqrt{6} \quad \|v_3\|=\sqrt{11}$

$$P=\begin{pmatrix}\frac{1}{\sqrt{66}}&\frac{1}{\sqrt{6}}&\frac{-3}{\sqrt{11}}\\\frac{-4}{\sqrt{66}}&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{11}}\\\frac{7}{\sqrt{66}}&\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{11}}\end{pmatrix}$$

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$$A=\begin{pmatrix}7&4&-4\\4&-8&-1\\-4&-1&-8\end{pmatrix}$$

$\lambda_1=9$

$$v_1=\begin{pmatrix}4\\1\\-1\end{pmatrix}$$

$\lambda_2=\lambda_3=-9$ repeated eigenvalue with multiplicity 2

Since A is symmetric, from this eigenvalue, we should get 2 linearly independent eigenvectors

For $\lambda_2=\lambda_3=-9$

$(A-\lambda I)x=\mathbf{0}\Rightarrow(A+9I)x=0$

$$\left(\begin{array}{ccc|c}16&4&-4&0\\4&1&-1&0\\-4&-1&1&0\end{array}\right)\xrightarrow{\frac{1}{16}R_1}\left(\begin{array}{ccc|c}1&\frac14&\frac{-1}{4}&0\\4&1&-1&0\\-4&-1&1&0\end{array}\right)\xrightarrow[4R_1+R_3]{-4R_1+R_2}\left(\begin{array}{ccc|c}1&\frac14&\frac{-1}{4}&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$x+\frac14 y-\frac14 z=0\Rightarrow x=\frac14 z-\frac14 y$

y is free, z is free } choose twice

choice 1: $z=y=1$

$$v_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

choice 2: chosen to be orthogonal to v_2

$$\begin{pmatrix}\frac14(z-y)\\y\\z\end{pmatrix}$$ to be an eigenvector

want:

$$\begin{pmatrix}\frac14(z-y)\\y\\z\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0$$ to be orthogonal

$\Rightarrow y+z=0$ choose $z=2,\ y=-2$

$$v_3=\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

We now have 3 linearly independent orthogonal eigenvectors, all that is left is to make them unit vectors.

$\|v_1\|=3\sqrt{2} \quad \|v_2\|=\sqrt{2} \quad \|v_3\|=\sqrt{17}$

$$P=\begin{pmatrix}\frac{4}{3\sqrt2}&0&\frac{1}{\sqrt{17}}\\\frac{1}{3\sqrt2}&\frac{1}{\sqrt2}&\frac{-2}{\sqrt{17}}\\\frac{-1}{3\sqrt2}&\frac{1}{\sqrt2}&\frac{2}{\sqrt{17}}\end{pmatrix}$$