8.12 Diagonalization

An $n \times n$ matrix A is **diagonalizable** if an invertible $n \times n$ matrix P can be found so that $P^{-1}AP = D$ is a diagonal matrix.

- The columns of P are the eigenvectors of A.
- The entries on the diagonal of D are the corresponding eigenvalues of A.

example:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \qquad \lambda_1 = 1, v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \lambda_2 = 5, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \qquad P^{-1}AP = D$$

Theorem: 8.12

An $n \times n$ matrix A is **diagonalizable** if and only if A has n linearly independent eigenvectors.

Theorem:

If an $n \times n$ matrix A has n distinct eigenvalues, then A is **diagonalizable**.

Note:
Not all diagonalizable matrices have n distinct eigenvalues, a matrix can be diagonalizable with a repeated eigenvalue.

What makes a matrix not diagonalizable is not the eigenvalues, it is the eigenvectors.

All diagonalizable matrices have n linearly independent eigenvectors.

example: **8.12**

$$A=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ -1 & -1 & 3\end{pmatrix}$$

$\lambda_1=\lambda_2=2, \lambda_3=3$

$\underline{\lambda_1=\lambda_2=2}$

$$(A-2I)\mathbf{v}=\mathbf{0}\Rightarrow\left(\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & -1 & 1 & 0\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}1 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$x=z-y$ Let $y=1$ and $z=0$ Let $y=0$ and $z=1$

y and z free

$$\Rightarrow v_1=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} \qquad \Rightarrow v_2=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$$

v_1 and v_2 are linearly independent

$\underline{\lambda_3=3}$

$$(A-3I)\mathbf{v}=\mathbf{0}\Rightarrow\left(\begin{array}{ccc|c}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ -1 & -1 & 0 & 0\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right)$$

$x=-y$ z is free

$x=0$ let $z=1$

$y=0$

$$\Rightarrow v_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$P=\begin{pmatrix}-1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix} \qquad D=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$$

8.12

An $n\times n$ matrix A is **orthogonally diagonalizable** if an orthogonal matrix P can be found so that $P^T AP=D$ is a diagonal matrix.

Theorem:

An $n\times n$ matrix A is **orthogonally diagonalizable** if and only if A is symmetric.

- The columns of P are the orthonormal eigenvectors of A.

One application of diagonalization is raising a matrix to a power.

Let A be diagonalizable.

$$P^{-1}AP = D$$

$$PP^{-1}AP = PD$$

$$AP = PD$$

$$APP^{-1} = PDP^{-1}$$

$$A = PDP^{-1}$$

$$A^2 = AA = \left(PDP^{-1}\right)\left(PDP^{-1}\right)$$

$$A^2 = PD\underbrace{PP^{-1}}_{I}DP^{-1}$$

$$A^2 = PDDP^{-1}$$

$$A^2 = PD^2P^{-1}$$

$$A^3 = AA^2 = \left(PDP^{-1}\right)\left(PD^2P^{-1}\right)$$

$$A^3 = PD\underbrace{PP^{-1}}_{I}D^2P^{-1}$$

$$A^3 = PDD^2P^{-1}$$

$$A^3 = PD^3P^{-1}$$

$$D^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

8.12

$A^n = PD^nP^{-1}$ n any integer

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \qquad P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^4 & 0 \\ 0 & 5^4 \end{pmatrix}\begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 625 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} -3 & 625 \\ 1 & 625 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} 628 & 1872 \\ 624 & 1876 \end{pmatrix}$$

$$= \begin{pmatrix} 157 & 468 \\ 156 & 469 \end{pmatrix}$$