Section 8.3
Rank of a Matrix

Let A be a $m \times n$ matrix.

The **rank** of A (denoted $\text{rank}(A)$) is the maximum number of linearly independent row vectors of A.

Let B be the row-echelon form of A.

$\boxed{\text{rank}(A) = \text{the number of nonzero rows of } B}$.

Example:
Find the rank of:

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{pmatrix} \begin{matrix} \\ -R_1 + R_2 \\ \\ -2R_1 + R_4 \end{matrix} \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 6 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 \end{pmatrix} R_3 \leftrightarrow R_2 \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 4 \\ 0 & 9 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ -6R_2 + R_3 \\ -9R_2 + R_4 \end{matrix} \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tfrac{1}{3}R_3$$

$$\boxed{Rank(A) = 3} \quad \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tfrac{4}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

technical definition:

Linear Independence

Let

a) V be a vector space.

b) $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}$ be vectors in V

c) Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$

If the only linear combination that gives the zero vector is when:

$c_1 = c_2 = \cdots = c_n = 0$,

then the set S is called **linearly independent**.

Said another way:

S is called **linearly independent** when

$c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n} = \mathbf{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$

If there are scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_n\mathbf{u_n} = \mathbf{0}$, then the set S is called **linearly dependent**

How you find whether a set of n vectors is linearly independent:

a) Make the vectors rows of a matrix.

b) Find the rank of the matrix.

rank of the matrix $= n$

$\Downarrow$

the n vectors are **linearly independent**

rank of the matrix $< n$

$\Downarrow$

the n vectors are **linearly dependent**

Solve : $A\mathbf{X} = \mathbf{B}$

Create : $\left(A \mid \mathbf{B}\right)$

$i)$ $\text{rank}\left(A\right) = \text{rank}\left(A \mid \mathbf{B}\right) \Rightarrow$ System is **consistent**

$$\begin{pmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$\text{rank}\left(A\right) = \text{rank}\left(A \mid \mathbf{B}\right) = n$

$\Rightarrow$ **Unique Solution**

$$\begin{pmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$\text{rank}\left(A\right) = \text{rank}\left(A \mid \mathbf{B}\right) < n$

$\Rightarrow$ **Infinitely Many Solutions**

$ii)$ $\text{rank}\left(A\right) < \text{rank}\left(A \mid \mathbf{B}\right) \Rightarrow$ System is **inconsistent**

$$\begin{pmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & 0 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \textbf{No Solution}$$