

Determinant

Every square matrix can be associated with a real number called the **determinant**.

Notation:

$$\det(A) \text{ or } |A|$$

Matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$\begin{vmatrix} -6 & 2 \\ -9 & 3 \end{vmatrix} = -18 - (-18) = 0$$

$$\begin{vmatrix} 0 & 3 \\ -1 & 99 \end{vmatrix} = 0 - (-3) = 3$$

Minor

Given a matrix A , each element of the matrix a_{ij} has a minor M_{ij} which is the determinant of the matrix obtained by deleting the i^{th} row and the j^{th} column of A .

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

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$$a_{11} = 1 \quad M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3$$

$$a_{12} = 2 \quad M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6$$

Cofactor

Each element a_{ij} also has a cofactor C_{ij} : $C_{ij} = (-1)^{i+j} M_{ij}$

$$C_{11} = (-1)^{1+1} M_{11} = -3 \quad i+j \text{ even} \Rightarrow \text{sign factor } +$$

$$C_{12} = (-1)^{1+2} M_{12} = 6 \quad i+j \text{ odd} \Rightarrow \text{sign factor } -$$

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Determinant Definition

Let A be a square matrix ($n \times n, n > 1$), the **determinant** is a real number that can be found using cofactor expansion on any row or column.

$$i^{\text{th}} \text{ row: } \det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}$$

(mult. each entry in the row by it's cofactor and find the sum of the products)

$$j^{\text{th}} \text{ column: } \det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj}$$

(mult. each entry in the col. by it's cofactor and find the sum of the products)

Section 8.4

Examples

Section 8.4

expanding on the first row:

$$\begin{vmatrix} 1 & -3 & 2 \\ -1 & 9 & 4 \\ -5 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 9 & 4 \\ 3 & 1 \end{vmatrix} + (-3)(-1) \begin{vmatrix} -1 & 4 \\ -5 & 1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 9 \\ -5 & 3 \end{vmatrix}$$

$$= 1(9-12) + 3(-1+20) + 2(-3+45) = -3 + 57 + 84 = 138$$

Take advantage of **zeros** when choosing a row or column to expand on

expanding on the third row:

$$\begin{vmatrix} 1 & -2 & 2 \\ 5 & -3 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 0 \cdot C_{31} + 0 \cdot C_{32} + 3 \begin{vmatrix} 1 & -2 \\ 5 & -3 \end{vmatrix} = 3(-3+10) = 21$$

3x3 Shortcut

Copy 1st 2 columns

$$\begin{vmatrix} 1 & 6 & -2 & 1 & 6 \\ 3 & -1 & 3 & 3 & -1 \\ 4 & 5 & 2 & 4 & 5 \end{vmatrix} = \left(\begin{matrix} \text{sum of} \\ \text{forward} \\ \text{diagonal} \\ \text{products} \end{matrix} \right) - \left(\begin{matrix} \text{sum of} \\ \text{backward} \\ \text{diagonal} \\ \text{products} \end{matrix} \right) = (-2 + 72 - 30) - (36 + 15 + 8) = 40 - 59 = -19$$

Triangular Matrices

$$A = \begin{pmatrix} 2 & -2 & -4 \\ 0 & -3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Upper Triangular

$$B = \begin{pmatrix} -3 & 0 & 0 \\ 7 & 4 & 0 \\ 88 & -11 & -5 \end{pmatrix}$$

Lower Triangular

$$C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Diagonal

Let T be a triangular matrix, $\det(T) =$ product of diagonal entries

$$\det(A) = 2(-3)(3) \Rightarrow \det(A) = -18$$

$$\det(B) = -3(4)(-5) \Rightarrow \det(B) = 60$$

$$\det(C) = -2(-1)(-4) \Rightarrow \det(C) = -8$$

Section 8.5

Using Elementary Row Operations to find the Determinant

Let B be obtained from A by:

a) multiplying a row by a number c (nonzero) $\det(A) = \frac{1}{c} \det(B)$

$$\begin{pmatrix} 2 & -1 & 2 & -5 \\ 0 & 3 & -1 & 5 \\ -5 & 8 & 11 & 3 \\ 4 & -9 & 6 & 2 \end{pmatrix} \xrightarrow{3R_1} \begin{pmatrix} 6 & -3 & 6 & -15 \\ 0 & 3 & -1 & 5 \\ -5 & 8 & 11 & 3 \\ 4 & -9 & 6 & 2 \end{pmatrix}$$

$\det(A) \qquad \det(B)$

b) Switch rows $\det(A) = -\det(B)$

$$\begin{pmatrix} 2 & -3 & 6 & -9 \\ 0 & 4 & -5 & 10 \\ 1 & 8 & 11 & 3 \\ 3 & -5 & 0 & -2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 8 & 11 & 3 \\ 0 & 4 & -5 & 10 \\ 2 & -3 & 6 & -9 \\ 3 & -5 & 0 & -2 \end{pmatrix}$$

$\det(A) \qquad \det(B)$

c) Add a multiple of one row to another row $\det(A) = \det(B)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{-4R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

$\det(A) \qquad \det(B)$

Row that is not changing Row you want to replace

Section 8.5

Goal:**a) Combine Row Reduction with expansion**

$$\begin{vmatrix} 1 & -2 & 7 & 9 \\ 3 & -4 & 5 & 5 \\ 3 & 6 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{vmatrix} \begin{matrix} -3R_1 + R_2 \\ -3R_1 + R_3 \\ -4R_1 + R_4 \end{matrix} = \begin{vmatrix} 1 & -2 & 7 & 9 \\ 0 & 2 & -16 & -22 \\ 0 & 12 & -20 & -28 \\ 0 & 13 & -25 & -34 \end{vmatrix} \begin{matrix} \frac{1}{2}R_2 \\ \frac{1}{4}R_3 \end{matrix} = (2)(4) \begin{vmatrix} 1 & -2 & 7 & 9 \\ 0 & 1 & -8 & -11 \\ 0 & 3 & -5 & -7 \\ 0 & 13 & -25 & -34 \end{vmatrix} \begin{matrix} \\ -3R_2 + R_3 \\ -13R_2 + R_4 \end{matrix} \\
 = (2)(4) \begin{vmatrix} 1 & -2 & 7 & 9 \\ 0 & 1 & -8 & -11 \\ 0 & 0 & 19 & 26 \\ 0 & 0 & 79 & 109 \end{vmatrix} = \underbrace{(2)(4)(1)(-1)^{1+1}}_8 \begin{vmatrix} 1 & -8 & -11 \\ 0 & 19 & 26 \\ 0 & 79 & 109 \end{vmatrix} = 8(1)(-1)^{1+1} \begin{vmatrix} 19 & 26 \\ 79 & 109 \end{vmatrix} = 8(19(109) - 79(26)) \\
 = 8(2071 - 2054) = 8(17) = \mathbf{136}$$

OR:**b) Row Reduce to Triangular Form**

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{matrix} -4R_1 + R_2 \\ -7R_1 + R_3 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} \begin{matrix} -\frac{1}{3}R_2 \\ -\frac{1}{6}R_3 \end{matrix} = -3(-6) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} \\ -R_2 + R_3 \end{matrix} = -3(-6) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} \\
 = -3(-6)(1)(1)(0) = \mathbf{0}$$

Properties of the Determinant

⊙ If elementary row (column) operations lead to one of the following three equivalent conditions, then the determinant is **zero**.

- an entire row (or column) consists of zeros
- two rows (or columns) are equal
- one row (or column) is a multiple of another row

Let A and B be $n \times n$, and let c be a scalar.

- ⊙ $\det(AB) = \det(A) \cdot \det(B)$
- ⊙ $\det(cA) = c^n \det(A)$
- ⊙ $\det(A^T) = \det(A)$