The Inverse of a Matrix

Section 8.6

A square $(n \times n)$ matrix A is **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I_{n\times n}$.

$I_{n\times n}$ is the $n \times n$ identity matrix.

B is called the (multiplicative) inverse of A.

The symbol used for B is A^{-1}.

A^{-1} is unique.

Not all matrices are invertible. A matrix that is not invertible is called **singular**.

example:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2(3)+5(-1) & 2(-5)+5(2) \\ 1(3)+3(-1) & 1(-5)+3(2) \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad AA^{-1} = I_{2\times 2}$$

You only need to check one direction since:

$AB = I$

$\underbrace{(AB)}_{I}A = A$ (mult. on rt. by A)

$A(BA) = A$ (mult. is assoc.)

$\Rightarrow BA = I$ (since $AI = A$)

Finding the Inverse of a 2 X 2 Matrix

Section 8.6

$$Let\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

❶ Switch a and d

❷ Negate b and c

❸ Calculate $D = ad - bc$

❹ Divide every entry by D.

$$A^{-1} = \begin{pmatrix} d/D & -b/D \\ -c/D & a/D \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 4/_{-2} & -2/_{-2} \\ -3/_{-2} & 1/_{-2} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

$D = ad - bc$

$D = 1(4) - 2(3)$

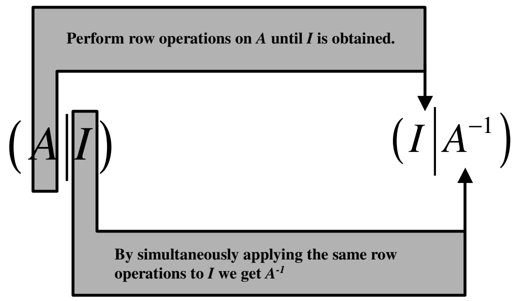
$D = -2$

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finding the Inverse of a 3 X 3 (or larger) Matrix (Method 1)

Let A be $n\times n$. Adjoin (attach) the $n\times n$ Identity matrix.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix.}}_{A}\underbrace{\left|\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{matrix}\right)}_{I}$$



<u>The 3 Elementary Row Operations</u> :

a) Multiply a row by a number (nonzero)

b) Switch rows

c) Add a multiple of $\underbrace{\text{one row}}_{\text{Row that is not changing}}$ to $\underbrace{\text{another row}}_{\text{Row you want to replace}}$

Section 8.6

Section 8.6

Let $A=\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$. Find A^{-1}.

$$\left(\underbrace{\begin{matrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{matrix}}_{A}\left|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right.\right) 2R_1 \Rightarrow \left(\begin{matrix} 4 & 0 & 2 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{matrix}\left|\begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.\right) R_3+R_1=NewR_1 \Rightarrow \left(\begin{matrix} -1 & 5 & 8 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{matrix}\left|\begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.\right) -R_1 \Rightarrow$$

$$\left(\begin{matrix} 1 & -5 & -8 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{matrix}\left|\begin{matrix} -2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.\right) \begin{matrix} 2R_1+R_2=NewR_2 \Rightarrow \\ 5R_1+R_3=NewR_3 \Rightarrow \end{matrix}$$

$$\Rightarrow + \begin{array}{r|rrrcrrr} 2R_1 & 2 & -10 & -16 & | & -4 & 0 & -2 \\ R_2 & -2 & 3 & 4 & | & 0 & 1 & 0 \\ \hline NewR_2 & 0 & -7 & -12 & | & -4 & 1 & -2 \end{array} \quad \Rightarrow + \begin{array}{r|rrrcrrr} 5R_1 & 5 & -25 & -40 & | & -10 & 0 & -5 \\ R_3 & -5 & 5 & 6 & | & 0 & 0 & 1 \\ \hline NewR_3 & 0 & -20 & -34 & | & -10 & 0 & -4 \end{array}$$

$$\left(\begin{matrix} 1 & -5 & -8 \\ 0 & -7 & -12 \\ 0 & -20 & -34 \end{matrix}\left|\begin{matrix} -2 & 0 & -1 \\ -4 & 1 & -2 \\ -10 & 0 & -4 \end{matrix}\right.\right) -3R_2 \Rightarrow \left(\begin{matrix} 1 & -5 & -8 \\ 0 & 21 & 36 \\ 0 & -20 & -34 \end{matrix}\left|\begin{matrix} -2 & 0 & -1 \\ 12 & -3 & 6 \\ -10 & 0 & -4 \end{matrix}\right.\right) R_3+R_2=NewR_2$$

$$\left(\begin{matrix} 1 & -5 & -8 \\ 0 & 1 & 2 \\ 0 & -20 & -34 \end{matrix}\left|\begin{matrix} -2 & 0 & -1 \\ 2 & -3 & 2 \\ -10 & 0 & -4 \end{matrix}\right.\right)$$

Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$. Find A^{-1}. (continued)

Section 8.6

$$\left(\begin{array}{ccc|ccc} 1 & -5 & -8 & -2 & 0 & -1 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & -20 & -34 & -10 & 0 & -4 \end{array}\right) \begin{array}{l} 5R_2 + R_1 = NewR_1 \Rightarrow \\ \\ 20R_2 + R_3 = NewR_3 \Rightarrow \end{array}$$

$$\begin{array}{r|cccccccc} 5R_2 & 0 & 5 & 10 & | & 10 & -15 & 10 \\ \Rightarrow + \quad R_1 & 1 & -5 & -8 & | & -2 & 0 & -1 \\ \hline NewR_1 & 1 & 0 & 2 & | & 8 & -15 & 9 \end{array} \qquad \begin{array}{r|cccccccc} 20R_2 & 0 & 20 & 40 & | & 40 & -60 & 40 \\ \Rightarrow + \quad R_3 & 0 & -20 & -34 & | & -10 & 0 & -4 \\ \hline NewR_3 & 0 & 0 & 6 & | & 30 & -60 & 36 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 8 & -15 & 9 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & 0 & 6 & 30 & -60 & 36 \end{array}\right) \tfrac{1}{6}R_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 8 & -15 & 9 \\ 0 & 1 & 2 & 2 & -3 & 2 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right) \begin{array}{l} -2R_3 + R_1 = NewR_1 \\ -2R_3 + R_2 = NewR_2 \\ \\ \end{array}$$

$$\begin{array}{r|cccccccc} -2R_3 & 0 & 0 & -2 & | & -10 & 20 & -12 \\ \Rightarrow + \quad R_1 & 1 & 0 & 2 & | & 8 & -15 & 9 \\ \hline NewR_1 & 1 & 0 & 0 & | & -2 & 5 & -3 \end{array} \qquad \begin{array}{r|cccccccc} -2R_3 & 0 & 0 & -2 & | & -10 & 20 & -12 \\ \Rightarrow + \quad R_2 & 0 & 1 & 2 & | & 2 & -3 & 2 \\ \hline NewR_2 & 0 & 1 & 0 & | & -8 & 17 & -10 \end{array}$$

$$\Bigg(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \Bigg| \underbrace{\begin{array}{ccc} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{array}}_{A^{-1}}\Bigg) \qquad A^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix} \qquad \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I}$$

Let $A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$. Find A^{-1}.

Section 8.6

$$\Bigg(\underbrace{\begin{array}{ccc} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{array}}_{A} \Bigg| \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I}\Bigg) \begin{array}{l} \\ -2R_1 + R_2 = NewR_2 \Rightarrow \\ -6R_1 + R_3 = NewR_3 \Rightarrow \end{array}$$

$$\begin{array}{r|cccccccc} -2R_1 & -2 & 2 & 4 & | & -2 & 0 & 0 \\ \Rightarrow + \quad R_2 & 2 & 4 & 5 & | & 0 & 1 & 0 \\ \hline NewR_2 & 0 & 6 & 9 & | & -2 & 1 & 0 \end{array} \qquad \begin{array}{r|cccccccc} -6R_1 & -6 & 6 & 12 & | & -6 & 0 & 0 \\ \Rightarrow + \quad R_3 & 6 & 0 & -3 & | & 0 & 0 & 1 \\ \hline NewR_3 & 0 & 6 & 9 & | & -6 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 6 & 9 & -6 & 0 & 1 \end{array}\right) -R_2 + R_3 = NewR_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right)$$

$$\begin{array}{r|cccccccc} -R_2 & 0 & -6 & -9 & | & 2 & -1 & 0 \\ \Rightarrow + \quad R_3 & 0 & 6 & 9 & | & -6 & 0 & 1 \\ \hline NewR_3 & 0 & 0 & 0 & | & -4 & -1 & 1 \end{array}$$

Further reduction will <u>always</u> yield another matrix with a row of zeros. This matrix can <u>never</u> be turned into I.

$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{pmatrix}$ is not invertible.

A is **<u>singular</u>**.

Properties of the Inverse

- $\left(A^{-1}\right)^{-1} = A$
- $\left(A^{k}\right)^{-1} = \left(A^{-1}\right)^{k}, \quad k > 0$
- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$
- $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$
- A is invertible if and only if $\det(A) \neq 0$.

Use the Inverse to Solve a System of Equations

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 1 \\ -2x_1 & + & 3x_2 & + & 4x_3 & = & -2 \\ -5x_1 & + & 5x_2 & + & 6x_3 & = & 3 \end{array} \Rightarrow \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}_{\mathbf{b}}$$

$A\vec{x} = \vec{b}$ and A is invertible

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

To solve the system for x, just find the inverse, and multiply it by b (in the order $A^{-1}b$)

Earlier we found $A^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}_{\mathbf{b}}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -21 \\ -72 \\ 43 \end{pmatrix}$$

Applications of the Determinant

a) An alternative method for finding the Inverse matrix :
Given an invertible matrix A, let C be the matrix of cofactors :

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

$$C^T = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} = adj(A), \text{ the } \textbf{adjoint} \text{ of } A$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Section 8.6

Section 8.6

◈ example of using the adjoint to find the inverse

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} \begin{vmatrix} 3 & 8 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 8 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix} \qquad C = \begin{pmatrix} -2 & -12 & 5 \\ 1 & 7 & -3 \\ 1 & 2 & -1 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -2 & 1 & 1 \\ -12 & 7 & 2 \\ 5 & -3 & -1 \end{pmatrix} = adj(A)$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} \\ -2R_1 + R_2 \\ R_1 + R_3 \end{matrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 3 & 7 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & -2 \\ 3 & 7 \end{vmatrix} \Rightarrow \det(A) = -1$$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = -1 \cdot \begin{pmatrix} -2 & 1 & 1 \\ -12 & 7 & 2 \\ 5 & -3 & -1 \end{pmatrix} \qquad \boxed{A^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{pmatrix}}$$

Elementary Matrices (Section 8.4)

A matrix obtained by performing a single elementary row operation to the identity matrix is called an **elementary matrix.**

The 3 Elementary Row Operations :

a) Multiply a row by a number (nonzero)

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} 3R_2 \Rightarrow \begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$$

b) Switch rows

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} R_2 \leftrightarrow R_4 \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$$

c) Add a multiple of $\underbrace{\text{one row}}_{\text{Row that is not changing}}$ to $\underbrace{\text{another row}}_{\text{Row you want to replace}}$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} 3R_2 + R_3 = NewR_3 \Rightarrow \begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}$$

Why does the method for finding the Inverse work?

$$A = \begin{pmatrix}0&1&3\\1&-3&0\\2&-6&2\end{pmatrix}$$

Perform row operations on A until you obtain I.
Keep track of each operation as an elementary matrix.

Elementary Matrix | **Result**

$$\begin{pmatrix}0&1&3\\1&-3&0\\2&-6&2\end{pmatrix} R_1 \leftrightarrow R_2 \quad E_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad E_1A = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&3\\1&-3&0\\2&-6&2\end{pmatrix} = \begin{pmatrix}1&-3&0\\0&1&3\\2&-6&2\end{pmatrix}$$

$$\begin{pmatrix}1&-3&0\\0&1&3\\2&-6&2\end{pmatrix} -2R_1 + R_3 \quad E_2 = \begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix} \quad E_2(E_1A) = \begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}1&-3&0\\0&1&3\\2&-6&2\end{pmatrix} = \begin{pmatrix}1&-3&0\\0&1&3\\0&0&2\end{pmatrix}$$

$$\begin{pmatrix}1&-3&0\\0&1&3\\0&0&2\end{pmatrix} 3R_2 + R_1 \quad E_3 = \begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix} \quad E_3(E_2E_1A) = \begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-3&0\\0&1&3\\0&0&2\end{pmatrix} = \begin{pmatrix}1&0&9\\0&1&3\\0&0&2\end{pmatrix}$$

$$\begin{pmatrix}1&0&9\\0&1&3\\0&0&2\end{pmatrix} \tfrac{1}{2}R_3 \quad E_4 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&\tfrac{1}{2}\end{pmatrix} \quad E_4(E_3E_2E_1A) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&\tfrac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&9\\0&1&3\\0&0&2\end{pmatrix} = \begin{pmatrix}1&0&9\\0&1&3\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&9\\0&1&3\\0&0&1\end{pmatrix} -9R_3 + R_1 \quad E_5 = \begin{pmatrix}1&0&-9\\0&1&0\\0&0&1\end{pmatrix} \quad E_5(E_4E_3E_2E_1A) = \begin{pmatrix}1&0&-9\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&9\\0&1&3\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix} -3R_3 + R_2 \quad E_6 = \begin{pmatrix}1&0&0\\0&1&-3\\0&0&1\end{pmatrix} \quad E_6(E_5E_4E_3E_2E_1A) = \begin{pmatrix}1&0&0\\0&1&-3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Why does the method for finding the Inverse work?

$$E_6E_5E_4E_3E_2E_1A = I$$

Start with A, the row operations that turn A into I are represented by $E_6E_5E_4E_3E_2E_1$.

$$\underbrace{E_6E_5E_4E_3E_2E_1}_{A^{-1}} A = I$$

$$A^{-1}A = I$$

$$A^{-1} = \left(E_6E_5E_4E_3E_2E_1\right) I$$

Start with I, the row operations that turn I into A^{-1} are represented by $E_6E_5E_4E_3E_2E_1$.