Application of the Determinant

Using Cramer's Rule to solve a system of equations

If a system of equations has a *n* x *n* coefficient matrix with nonzero determinant, then the solution can be found by the following method:

$$Ax = b$$

$$\begin{pmatrix} a_{11} & a_{2n} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Make a new matrix A_1 where the 1^{st} column of A is replaced by b.

Make a new matrix A_2 where the 2^{nd} column of A is replaced by b.

Continue this process up to A_n.

$$A_1 = \begin{pmatrix} b_1 & a_{2n} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad A_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix} \cdots A_n = \begin{pmatrix} a_{11} & a_{2n} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

The variables $x_1, x_2, \ldots, x_n$ can be found by the following :

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)} \quad , \ldots , x_n = \frac{\det(A_n)}{\det(A)}$$

◈ example of using Cramer's Rule to solve a system of equations

$$\begin{array}{ccccccccc} 2x_1 & + & 4x_2 & - & x_3 & = & -1 \\ 4x_1 & - & 2x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 4 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$\det(A) = 2(-2-1) - 4(4-1) - 1(4+2)$$

$$\det(A) = -6 - 12 - 6 = -24$$

$$A_1 = \begin{pmatrix} -1 & 4 & -1 \\ 9 & -2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\det(A_1) = -1\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 9 & 1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 9 & -2 \\ 3 & 1 \end{vmatrix}$$

$$\det(A_1) = -1(-2-1) - 4(9-3) - 1(9+6)$$

$$\det(A_1) = 3 - 24 - 15 = -36$$

$$A_2 = \begin{pmatrix} 2 & -1 & -1 \\ 4 & 9 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

$$\det(A_2) = 2\begin{vmatrix} 9 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 9 \\ 1 & 3 \end{vmatrix}$$

$$\det(A_2) = 2(9-3) + 1(4-1) - 1(12-9)$$

$$\det(A_2) = 12 + 3 - 3 = 12$$

$$A_3 = \begin{pmatrix} 2 & 4 & -1 \\ 4 & -2 & 9 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\det(A_3) = 2\begin{vmatrix} -2 & 9 \\ 1 & 3 \end{vmatrix} - 4\begin{vmatrix} 4 & 9 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$\det(A_3) = 2(-6-9) - 4(12-9) - 1(4+2)$$

$$\det(A_3) = -30 - 12 - 6 = -48$$

$$x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_1 = \frac{-36}{-24}$$

$$\Rightarrow \boxed{x_1 = \frac{3}{2}}$$

$$x_2 = \frac{\det(A_2)}{\det(A)}$$

$$x_2 = \frac{12}{-24}$$

$$\Rightarrow \boxed{x_2 = -\frac{1}{2}}$$

$$x_3 = \frac{\det(A_3)}{\det(A)}$$

$$x_3 = \frac{-48}{-24}$$

$$\Rightarrow \boxed{x_3 = 2}$$