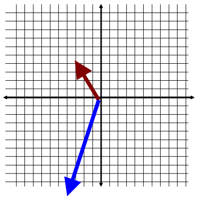
Section 8.8 Eigenvalues and Eigenvectors

Matrix-vector multiplication can be thought of geometrically as a linear transformation changing one vector into another

$$\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -11 \end{pmatrix}$$



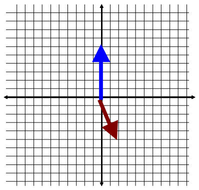
$$\begin{matrix} 4x & + & 2y & = & 0 \\ 5x & + & y & = & 6 \end{matrix}$$

$$\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

What vector **x** gets transformed into the vector **b**?

$$\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

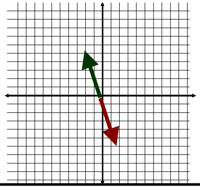


$$A\mathbf{x} = \lambda\mathbf{x}$$

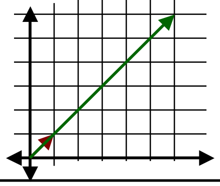
eigenvector **eigenvalue**

What vector **x** gets transformed into $\lambda\mathbf{x}$ (a scalar mult. of the orig. **x**)?

$$\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Section 8.8 Eigenvalues and Eigenvectors

Solve:

$$Ax = \lambda x \quad (A_{n\times n})$$

$$Ax - \lambda x = \mathbf{0}$$

$$\underbrace{(A - \lambda I)}_{\text{matrix}}\underset{\text{vector}}{\underset{\uparrow}{x}} = \underset{\text{vector}}{\underset{\uparrow}{\mathbf{0}}}$$

Need $(A - \lambda I)$ to not be invertible, because if it was we would only have the trivial solution $x = 0$.

$$\Downarrow$$

Set $\det(A - \lambda I) = 0$

This leads to an equation in λ called the **characteristic equation**.

The roots of the characteristic equation are the eigenvalues λ.

For each eigenvalue λ, find its eigenvector by solving $(A - \lambda I)x = \mathbf{0}$

Find the eigenvalues of $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$.

$\det(A-\lambda I)=0 \Rightarrow$

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} + (-1)1\begin{vmatrix} 1 & 1 \\ 0 & -1-\lambda \end{vmatrix}$$

$$=(-1-\lambda)\left[(2-\lambda)(-1-\lambda)-3\right]-(-1-\lambda)=0$$

$$=(-1-\lambda)\left[\left(-2-\lambda+\lambda^2\right)-3\right]-(-1-\lambda)=0$$

$$=(-1-\lambda)\left[\lambda^2-\lambda-5\right]-(-1-\lambda)=0$$

$$=(-1-\lambda)\left(\left(\lambda^2-\lambda-5\right)-1\right)=0$$

$$=(-1-\lambda)\left(\lambda^2-\lambda-6\right)=0$$

$$=(-1-\lambda)(\lambda-3)(\lambda+2)=0$$

$$\boxed{\lambda_1=-1, \lambda_2=-2, \lambda_3=3}$$

There must be an easier way ☺

Shortcut to finding the characteristic equation

2×2

$$\lambda^2-\underbrace{\left(trace(A)\right)}_{\text{sum of the diagonal entries}}\lambda+\det(A)=0$$

3×3

$$\lambda^3-\left(trace(A)\right)\lambda^2+\underbrace{\left(C_{11}+C_{22}+C_{33}\right)}_{\text{sum of the diagonal cofactors}}\lambda-\det(A)=0$$

The only problem now is that you have to factor a cubic ☹

Find one root, then use synthetic (or long) division to get the quadratic factor, then factor that (if possible)

$n \times n$ triangular or diagonal matrix

The eigenvalues are on the diagonal!

Find the eigenvalues of $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

using the 3×3 shortcut

$$\lambda^3 - (trace(A))\lambda^2 + (C_{11}+C_{22}+C_{33})\lambda - \det(A) = 0$$

$$trace(A) = -1+2-1 = 0$$

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2-3 = -5 \qquad C_{22} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \qquad C_{33} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -2-1 = -3 \quad \Rightarrow C_{11}+C_{22}+C_{33} = -7$$

$$\det(A) = -1\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + (1)(-1)\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1(-2-3)-1(-1) = 6$$

Characteristic Equation:

$$\lambda^3 - 0\cdot\lambda^2 - 7\lambda - 6 = 0 \Rightarrow \lambda^3 - 7\lambda - 6 = 0$$

Plug these in until you find all the roots.

Possible roots:

$\pm 1, \pm 2, \pm 3,$ and ± 6 (since the constant term is -6)

$\lambda^3 - 7\lambda - 6 = 0$

$\lambda = 1? \Rightarrow 1-7-6 \neq 0$ No

$\lambda = -1? \Rightarrow -1+7-6 = 0$ Yes

$\lambda = 2? \Rightarrow 8-14-6 \neq 0$ No

$\lambda = -2? \Rightarrow -8+14-6 = 0$ Yes

$\lambda = 3? \Rightarrow 27-21-6 = 0$ Yes

$$\boxed{\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 3}$$

Or you could find one root, then use synthetic (or long) division to find the quadratic factor.

Synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & \downarrow & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$\Rightarrow \lambda^3 - 7\lambda - 6 = (\lambda+1)(\lambda^2-\lambda-6)$$

$$\lambda^3 - 7\lambda - 6 = 0 \Rightarrow (\lambda+1)(\lambda+2)(\lambda-3) = 0$$

Long division:

$$\begin{array}{r} \lambda^2 - \lambda - 6 \\ \lambda+1 \overline{\big) \lambda^3 + 0\lambda^2 - 7\lambda - 6} \\ \underline{-(\lambda^3 + \lambda^2)} \\ -\lambda^2 - 7\lambda \\ \underline{-(-\lambda^2 - \lambda)} \\ -6\lambda - 6 \\ \underline{-(-6\lambda - 6)} \\ 0 \end{array}$$

Find the eigenvectors of $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

Eigenvalues:

$$\boxed{\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 3}$$

For $\lambda_1 = -1$

$(A-\lambda I)x = \mathbf{0} \Rightarrow (A+I)v_1 = 0$

z is free, let $z = 1$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) -3R_2 + R_3 \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x + 3y + z = 0 \Rightarrow x = -z$

$\Rightarrow y = 0$

$$\boxed{v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2$

$(A-\lambda I)x = \mathbf{0} \Rightarrow (A+2I)v_2 = 0$

z is free, let $z = 3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) -R_1 + R_2 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) -R_2 + R_3 \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x + y = 0 \Rightarrow x = -y \Rightarrow x = \frac{1}{3}z$

$3y + z = 0 \Rightarrow y = \frac{-1}{3}z$

$$\boxed{v_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}$$

For $\lambda_3 = 3$

$(A-\lambda I)x = \mathbf{0} \Rightarrow (A-3I)v_3 = 0$

$x - y + z = 0 \Rightarrow x = y - z \Rightarrow x = \frac{1}{3}z$

$$\left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) 4R_1 + R_2 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right) R_2 + R_3 \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-3y + 4z = 0 \Rightarrow y = \frac{4}{3}z$

z is free, let $z = 3$

$$\boxed{v_3 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}}$$

8.8 # 18

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow$

$$\begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(2-\lambda)-1\right] = 0$$

$$(1-\lambda)\left[4-4\lambda+\lambda^2-1\right] = 0$$

$$(1-\lambda)\left[\lambda^2-4\lambda+3\right] = 0 \Rightarrow (1-\lambda)(\lambda-1)(\lambda-3) = 0$$

$$\boxed{\lambda_1 = \lambda_2 = 1, \lambda_3 = 3}$$

For $\lambda_1 = \lambda_2 = 1$

$(A - I)v_1 = 0$

$$\left(\begin{array}{ccc|c} 0 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) -R_3 + R_2 \left(\begin{array}{ccc|c} 0 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) -6R_1 + R_2 \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} y = -z \Rightarrow y = 0 \\ \Rightarrow z = 0 \end{array}$$

x is free, let $x = 1$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3$

$(A - 3I)v_3 = 0$

$$\left(\begin{array}{ccc|c} -2 & 6 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \tfrac{-1}{2}R_1 \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x = 3y \Rightarrow x = 3z \\ \Rightarrow y = z \end{array}$$

z is free, let $z = 1$

$$\boxed{v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}$$

8.8 # 14

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$.

$\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$ is a diagonal matrix (its eigenvalues are on the diagonal)

$$\boxed{\lambda_1 = 7 \text{ and } \lambda_2 = 13}$$

For $\lambda_1 = 7$

$(A - 7I)v_1 = 0$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 6 & 0 \end{array}\right) \begin{array}{l} x \text{ is free,let } x = 1 \\ \Rightarrow y = 0 \end{array}$$

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

For $\lambda_2 = 13$

$(A - 13I)v_2 = 0$

$$\left(\begin{array}{cc|c} -6 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow x = 0 \\ y \text{ is free,let } y = 1 \end{array}$$

$$\boxed{v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

The eigenvectors of a diagonal matrix will be the columns of the identity matrix.