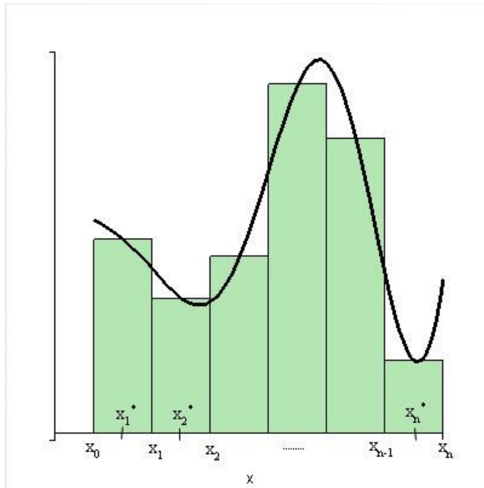
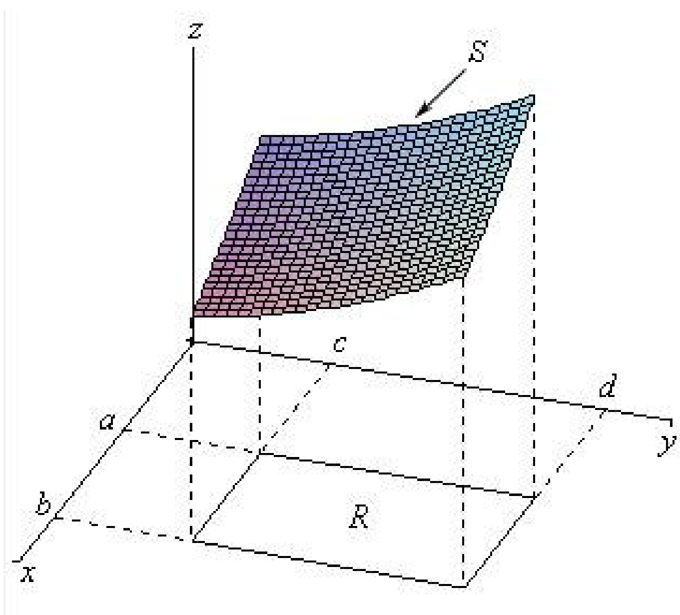
9.10 Double Integrals



$A \approx \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ $\qquad$ $A = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$ $\qquad$ $A = \int_a^b f(x)\, dx$



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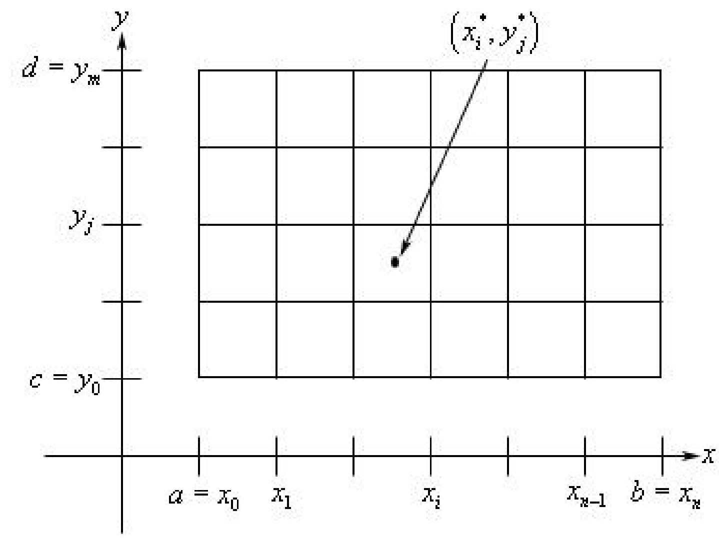

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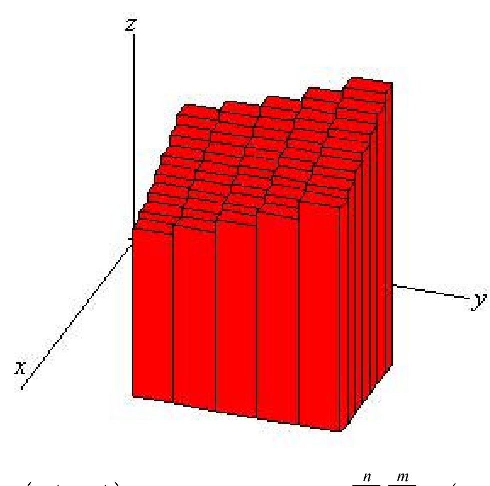

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A \qquad V = \lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i^*, y_j^*\right)\Delta A$$

$$V = \int_c^d\int_a^b f(x, y)\, dxdy = \int_a^b\int_c^d f(x, y)\, dydx$$

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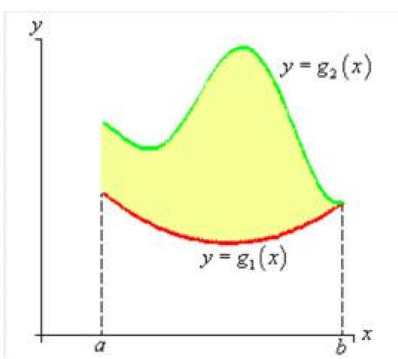
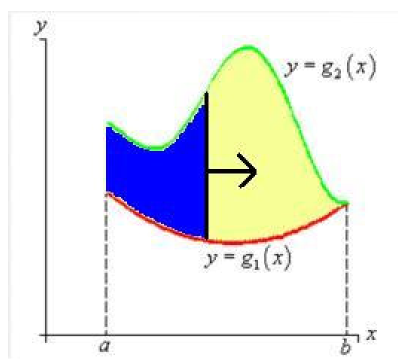
$$\int_0^1 \left(\int_0^2 4 - x - y \, dx \right) dy = \int_0^1 \left(4x - \frac{x^2}{2} - xy \Big|_0^2 \right) dy = \int_0^1 \left[(8 - 2 - 2y) - 0 \right] dy$$

$$= \int_0^1 6 - 2y \, dy = 6y - y^2 \Big|_0^1 = 5$$

9.10

Double Integrals over general regions





$dydx$

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx$$

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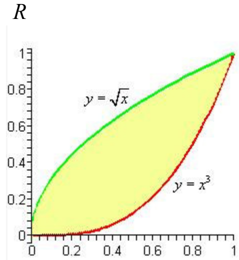
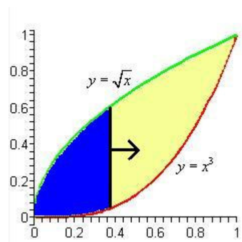
$$\iint_R 4xy - y^3\,dA \quad R = \text{the region between } y = \sqrt{x} \text{ and } y = x^3$$





$$\iint_R 4xy - y^3\,dydx$$

$$\int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\,dydx = \int_0^1 \left[2xy^2 - \frac{y^4}{4}\right]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \underbrace{\left(2x^2 - \frac{x^2}{4}\right)}_{\frac{7x^2}{4}} - \left(2x^7 - \frac{x^{12}}{4}\right) dx = \int_0^1 \left(\frac{7x^2}{4} - 2x^7 + \frac{x^{12}}{4}\right) dx = \left(\frac{7x^3}{12} - \frac{x^8}{4} + \frac{x^{13}}{52}\right)\Bigg|_0^1$$

$$= \frac{7}{12} - \frac{1}{4} + \frac{1}{52} = \frac{1}{3} + \frac{1}{52} = \boxed{\frac{55}{156}}$$

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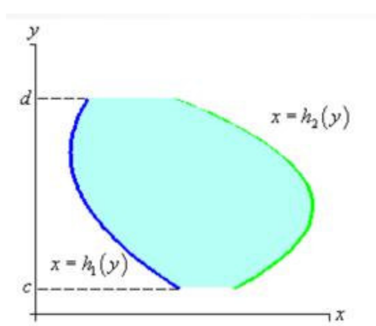

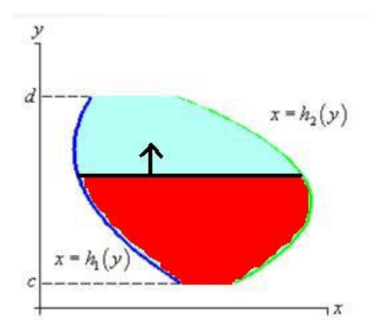

$$dxdy$$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

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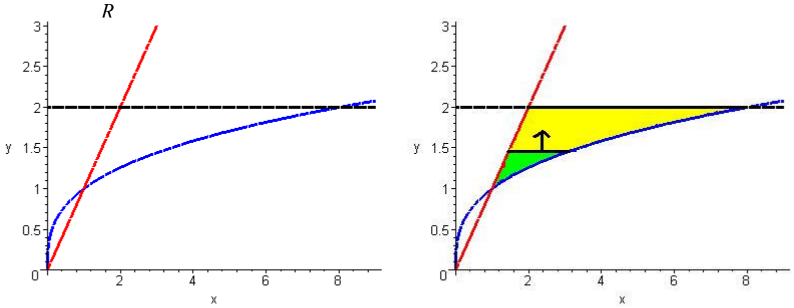
$$\iint_R e^{x/y}\,dA \quad R = \text{the region } 1 \le y \le 2,\ y^3 \le x \le y$$



$$\iint_R e^{x/y}\,dxdy$$

$$= \int_1^2 \int_y^{y^3} e^{x/y}\,dxdy = \int_1^2 \left[y e^{x/y} \right]_y^{y^3} dy = \int_1^2 \left(y e^{y^2} - y e^1 \right) dy$$

$$= \left[\frac{1}{2} e^{y^2} - \frac{y^2}{2} e^1 \right]_1^2 = \left(\frac{1}{2} e^4 - 2e \right) - \left(\frac{1}{2} e - \frac{1}{2} e \right) = \boxed{\frac{1}{2} e^4 - 2e}$$

Area of a 2-dimensional region using double integration

Use $f(x,y) = 1$ $$\iint_R dA$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dydx = \int_a^b \left[g_2(x) - g_1(x) \right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} dxdy = \int_c^d \left[h_2(y) - h_1(y) \right] dy$$

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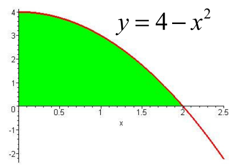

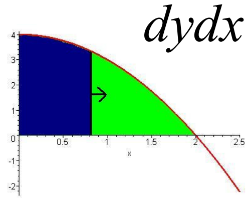

$dydx$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dydx = \int_0^2 \int_0^{4-x^2} dydx = \int_0^2 \left[4-x^2\right]dx$$

$$=\left[4x-\frac{x^3}{3}\right]_0^2 = 8-\frac{8}{3} = \boxed{\frac{16}{3}}$$

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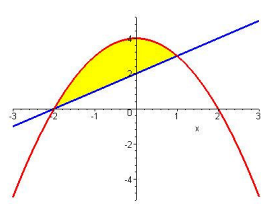
Order of Integration dictated by:

a) the integrand

$$\int_0^2 \int_x^2 e^{-y^2}\,dydx$$

b) the region

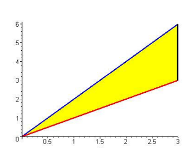
$$\iint_R dA \quad R: x+2 \le y \le 4-x^2$$



c) Both the integrand and the region

$$\iint_R \frac{y}{x^2+y^2} \quad R:\ \text{triangle bounded}$$

by $y=x, y=2x, x=3$



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Double Integral Applications

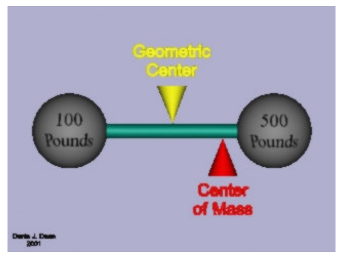
Center of Mass

$$mass = \iint_R \rho(x,y)\,dA$$

$$M_x = \iint_R y\rho(x,y)\,dA \quad M_y = \iint_R x\rho(x,y)\,dA$$

$$Center\ of\ Mass = (\bar{x},\bar{y}) = \left(\frac{M_y}{mass},\frac{M_x}{mass}\right)$$

1st moments : "balancing" moments



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Inertia **2nd moments : "turning" moments**

The moment of inertia of an object about a given axis describes how difficult it is to change its angular motion about that axis

$$I_x = \iint_R y^2\rho(x,y)\,dA \quad I_y = \iint_R x^2\rho(x,y)\,dA$$

Moment of Inertia about $z-axis = I_0 = I_x + I_y$ (polar moment)

$$I_0 = \iint_R (x^2+y^2)\rho(x,y)\,dA = I_0 = \iint_R r^2\rho(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Radius of gyration about $y-axis$: $\bar{\bar{x}} = \sqrt{\frac{I_y}{mass}}$

Radius of gyration about $x-axis$: $\bar{\bar{y}} = \sqrt{\frac{I_x}{mass}}$

How far from the axis the entire mass might be concentrated to give the same inertia

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