9.11 Double Integrals in Polar Coordinates

Change of variables

$x = f(u,v), \quad y = g(u,v)$

Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_R F(x,y)\,dA = \iint_S F(f(u.v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

absolute value of the Jacobian

Change of variables into Polar Coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

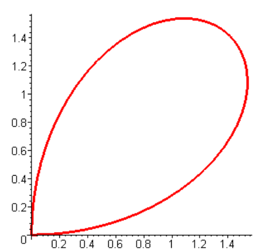
$$\iint_R F(x,y)\boxed{dA} = \iint_S F(r\cos\theta, r\sin\theta)\boxed{rdrd\theta}$$

Area of a region in Polar Coordinates

Area of Region $R = \iint_R dA \;= \iint_R rdrd\theta$ in polar coordinates

Find the area enclosed by the graph of

$r = 2\sin 2\theta \quad 0 \le \theta \le \frac{\pi}{2}$



$$Area = \int_0^{\pi/2}\int_0^{2\sin 2\theta} rdrd\theta \;= \int_0^{\pi/2}\left(\frac{1}{2}r^2\Big|_0^{2\sin 2\theta}\right)d\theta$$

$$= 2\int_0^{\pi/2}\sin^2 2\theta d\theta \;= \int_0^{\pi/2}(1-\cos 4\theta)\,d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \Rightarrow \sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$

$$= \left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2} \;= \boxed{\frac{\pi}{2}}$$

Evaluate the given integral by converting to polar coordinates. **9.11**

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dydx$$

$e^{x^2+y^2} \to e^{r^2}$, $\quad dydx \to rdrd\theta$

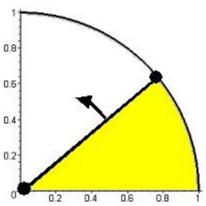
Find the region:

$0 \le y \le \sqrt{1-x^2}$

$0 \le x \le 1$

$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1$

Circle of radius 1, but only the first quadrant since $y \ge 0$ and $0 \le x \le 1$.



The region in polar:

$0 \le r \le 1$

$0 \le \theta \le \dfrac{\pi}{2}$

In polar coords.:

$$\int_0^{\pi/2}\int_0^1 e^{r^2}\, rdrd\theta = \int_0^{\pi/2}\left(\frac{1}{2}e^{r^2}\right)\Bigg|_0^1 d\theta = \int_0^{\pi/2}\frac{1}{2}(e-1)\,d\theta = \frac{1}{2}(e-1)\int_0^{\pi/2} d\theta$$

$u = r^2$

$du = 2rdr \quad \frac{1}{2}du = rdr$

$\frac{1}{2}\int e^u du = \frac{1}{2}e^u \to \frac{1}{2}e^{r^2}$

$$= \boxed{\frac{\pi}{4}(e-1)}$$

9.11

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dydx$$

$\dfrac{2}{1+\sqrt{x^2+y^2}} \to \dfrac{2}{1+r}$, $\quad dydx \to rdrd\theta$

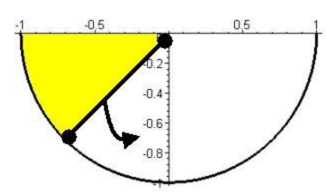
Find the region:

$-\sqrt{1-x^2} \le y \le 0$

$-1 \le x \le 1$

$y = -\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1$

Circle of radius 1, but only the third and fourth quadrants since $y \le 0$ and $-1 \le x \le 1$.



The region in polar:

$0 \le r \le 1$

$\pi \le \theta \le 2\pi$

In polar coords.:

$$\int_{\pi}^{2\pi}\int_0^1 \frac{2r}{1+r}\, drd\theta$$

$u = 1+r, \quad du = dr, \quad r = u-1$

$\int \frac{2(u-1)}{u}du = \int \frac{2u-2}{u}du = \int 2 - \frac{2}{u}du = 2\int\left(1-\frac{1}{u}\right)du$

$= 2(u - \ln u) \Rightarrow 2((1+r) - \ln(1+r))$

$$= \int_{\pi}^{2\pi} 2\left((1+r)-\ln(1+r)\right)\Big|_0^1 d\theta$$

$$= 2\left[(2-\ln 2)-1\right]\int_{\pi}^{2\pi} d\theta = \boxed{2\pi(1-\ln 2)}$$

Evaluate the given integral by converting to polar coordinates **9.11**

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$\sqrt{x^2+y^2} \to r$, $dydx \to rdrd\theta$

Find the region:

$0 \le y \le \sqrt{2x-x^2}$

$0 \le x \le 2$

$y = \sqrt{2x-x^2}$ since $y = r\sin\theta$ and $x = r\cos\theta$, then

$$r\sin\theta = \sqrt{2r\cos\theta - (r\cos\theta)^2}$$

$$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$$r^2 = 2r\cos\theta \quad \text{(don't divide by } r, \text{ it could be zero)}$$

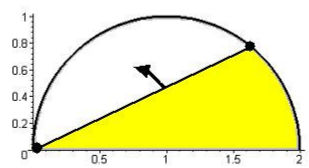
$$r^2 - 2r\cos\theta = 0$$

$$r(r - 2\cos\theta) = 0 \Rightarrow \text{either } r = 0 \text{ or } r = 2\cos\theta$$

These are the bounds on r, but what about θ?

Since $x \ge 0$ and $y \ge 0$, the region must be in quadrant I.

$\Rightarrow 0 \le \theta \le \frac{\pi}{2}$



$$\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta = \int_0^{\pi/2} \left(\frac{1}{3}r^3\right)\Big|_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{8}{3}\int_0^{\pi/2} \cos\theta\cos^2\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos\theta\left(1-\sin^2\theta\right) d\theta$$

$u = \sin\theta,\ du = \cos\theta d\theta$

$\int\left(1-u^2\right) du = u - \frac{1}{3}u^3$

$\Rightarrow \sin\theta - \frac{1}{3}\sin^3\theta$

$$= \frac{8}{3}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right)\Big|_0^{\pi/2} = \frac{8}{3}\left(1 - \tfrac{1}{3}\right)$$

$$= \boxed{\frac{16}{9}}$$