Section 9.13 Surface Integrals

Consider the surface given by $z = f(x, y)$.

Let R be the projection of the surface onto the xy plane.

$$\text{Surface Area} = \iint_R \sqrt{1 + f_x(x, y) + f_y(x, y)}\ dA$$

$\left(\text{provided } f_x \text{ and } f_y \text{ are continuous on } R\right)$

$$\iint_R f(x, y)\, dA \overset{f(x,y)=1}{\Leftrightarrow} \iint_R dA$$

INTEGRAL — **Area of Region R**

$$\iint_R G\Big(x, y, \underbrace{f(x, y)}_{z}\Big)\sqrt{1 + f_x(x, y) + f_y(x, y)}\ dA \overset{G(x,y,z)=1}{\Leftrightarrow} \iint_R \sqrt{1 + f_x(x, y) + f_y(x, y)}\ dA$$

Surface Integral — **Surface Area**

Surface Integral

$$\iint_S G(x, y, z)\, dS = \iint_R G\Big(x, y, \underbrace{f(x, y)}_{z}\Big)\sqrt{1 + f_x(x, y) + f_y(x, y)}\ dA$$

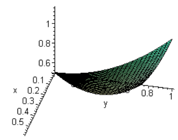
R – projection of S onto the xy plane

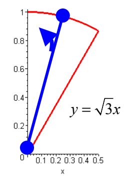
G, f, f_x, f_y need to be continuous

throughout the region containing S.

Section 9.13 #22

Evaluate the surface integral $\iint_S G(x,y,z)\,dS$ with $G(x,y,z)=2z$ and S is the portion of the paraboloid $2z=1+x^2+y^2$ in the first octant bounded by $x=0, y=\sqrt{3}x,$ and $z=1$.





$$z=\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2}y^2=f(x,y)$$

$$\iint_S G(x,y,z)\sqrt{1+(f_x)^2+(f_y)^2}\,dA$$

$$\iint_S G(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\,dA$$

$$\iint_S 2\left(\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2}y^2\right)\sqrt{1+x^2+y^2}\,dA$$

$$\int_{\pi/3}^{\pi/2}\int_0^1 (1+r^2)\sqrt{1+r^2}\,r\,dr\,d\theta = \int_{\pi/3}^{\pi/2}\int_0^1 (1+r^2)^{3/2}\,r\,dr\,d\theta$$

$$=\int_{\pi/3}^{\pi/2}\left(\frac{1}{5}(1+r^2)\Big|_0^1\right)d\theta = \frac{1}{5}\left(2^{5/2}-1\right)\int_{\pi/3}^{\pi/2}d\theta = \boxed{\frac{\pi}{30}\left(4\sqrt{2}-1\right)}$$

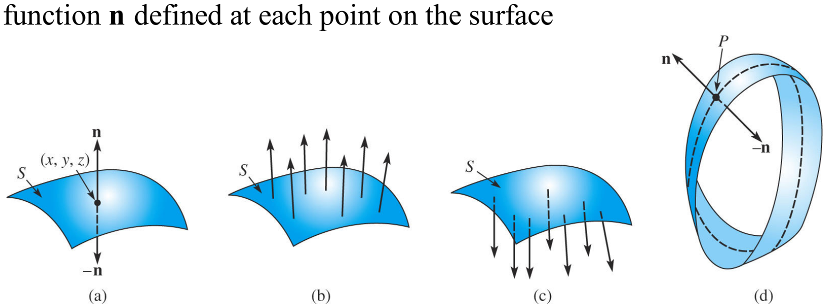
$u=1+r^2$

$du=2r\,dr \qquad \frac{1}{2}du=r\,dr$

$\frac{1}{2}\int u^{3/2}du=\frac{1}{2}\cdot\frac{2}{5}u^{5/2}$

Surface Integrals of Vector Fields

A surface S is **orientable** if there exists a continuous unit normal vector function $\mathbf{n}$ defined at each point on the surface



(a) orientable surface with $\mathbf{n}$ and $-\mathbf{n}$

(b) upward orientation $\mathbf{n}$

(c) downward orientation $-\mathbf{n}$

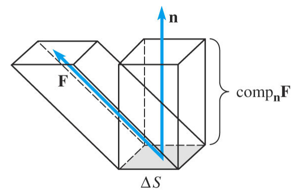
(d) non-orientable surface since $\mathbf{n}$ becomes $-\mathbf{n}$

To find $\mathbf{n}$, define the surface S by $g(x,y,z)=0$, then $\mathbf{n}=\left(\dfrac{1}{\|\nabla g\|}\right)\nabla g$

Let $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$ be the velocity field of a fluid.

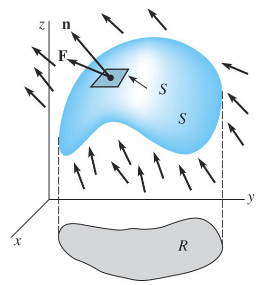
Volume of fluid flowing through
an element of surface area ΔS per unit time
is approximated by

$$(\text{comp}_{\mathbf{n}}\mathbf{F})\Delta S=(\mathbf{F}\cdot\mathbf{n})\,dS$$



The total volume of fluid passinging through the surface S per unit time is called the **flux of F through S**.

$$\text{flux}=\iint_S(\mathbf{F}\cdot\mathbf{n})\,dS$$



Section 9.13 #33

$\mathbf{F}=\langle\frac{1}{2}x^2,\frac{1}{2}y^2,z\rangle$ S : the portion of the paraboloid $z=4-x^2-y^2$ $0\le z\le 4$

Compute flux of $\mathbf{F}$ through S

$$g=x^2+y^2+z-4$$

$$\nabla g=\langle 2x,2y,1\rangle$$

$$\|\nabla g\|=\sqrt{4x^2+4y^2+1}$$

$$\mathbf{n}=\left(\frac{1}{\|\nabla g\|}\right)\nabla g=\frac{1}{\sqrt{4x^2+4y^2+1}}\langle 2x,2y,1\rangle$$

$$\mathbf{F}\cdot\mathbf{n}=\frac{x^3+y^3+z}{\sqrt{4x^2+4y^2+1}}$$

$$dS=\sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

$$dS=\sqrt{1+4x^2+4y^2}\,dA$$

$$\text{flux}=\iint_S(\mathbf{F}\cdot\mathbf{n})\,dS$$

$$\text{flux}=\iint_R\frac{x^3+y^3+z}{\cancel{\sqrt{4x^2+4y^2+1}}}\cancel{\sqrt{1+4x^2+4y^2}}\,dA$$

$$\text{flux}=\iint_R\left(x^3+y^3+\left(4-x^2-y^2\right)\right)dA$$

$$z=0\Rightarrow x^2+y^2=4$$

$$\Rightarrow 0\le r\le 2, 0\le\theta\le 2\pi$$

$$=\int_0^{2\pi}\int_0^2\left(r^3\left(\cos^3\theta+\sin^3\theta\right)+4-r^2\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(\tfrac{r^5}{5}\left(\cos^3\theta+\sin^3\theta\right)+2r^2-\tfrac{r^4}{4}\right)\Big|_0^2\,d\theta$$

$$=\int_0^{2\pi}\left(\tfrac{32}{5}\left(\cos^3\theta+\sin^3\theta\right)+4\right)d\theta=\boxed{8\pi}$$