

9.14 Stokes' Theorem

- S : a piecewise smooth orientable surface
- C : a piecewise smooth simple closed curve that bounds S , oriented counter-clockwise as viewed from above
- \mathbf{n} : the unit normal to S , defines orientation of S
- \mathbf{F} : $\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is a vector field with P, Q, R , and all first partial derivatives continuous in a region of 3-space containing S

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

3-dim line \Leftrightarrow surface

integral integral

Use Stokes' Theorem to evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$

By finding

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r}} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

Evaluate

Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$

Evaluate **By finding**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS}$$

Example 1 Use Stokes' Theorem to evaluate

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS \quad \text{when } \mathbf{F} = \langle y^2 z, xz, x^2 y^2 \rangle \text{ and } S \text{ is}$$

that part of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 1$, oriented upward.

$$\Rightarrow \text{Find } \oint_C \mathbf{F} \cdot d\mathbf{r} \quad \left. \begin{array}{l} x = \cos t \\ y = \sin t \\ z = 1 \end{array} \right\} 0 \leq t \leq 2\pi$$

Parametrize C :

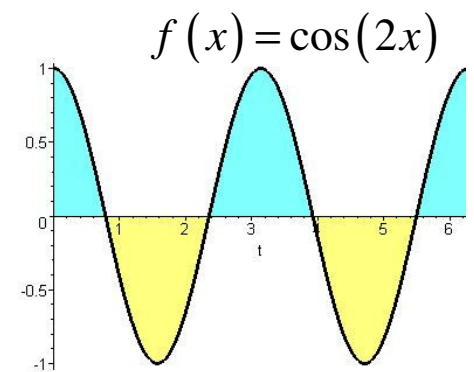
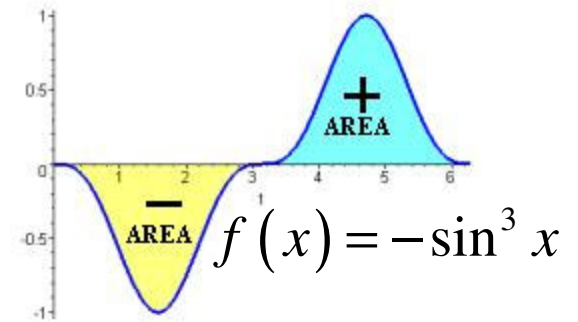
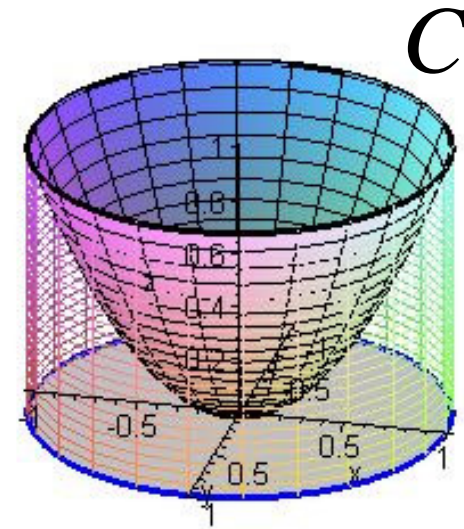
$$\mathbf{r} = \langle \cos t, \sin t, 1 \rangle \Rightarrow d\mathbf{r} = \langle -\sin t, \cos t, 0 \rangle dt$$

$$\mathbf{F} \text{ on } C : \langle \sin^2 t, \cos t, \cos^2 t \sin^2 t \rangle$$

$$\mathbf{F} \cdot d\mathbf{r} = (-\sin^3 t + \cos^2 t) dt$$

$$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^3 t + \cos^2 t) dt$$

$$= \int_0^{2\pi} \cos^2 t \, dt = \frac{1}{2} \int_0^{2\pi} (1 + \cos(2t)) \, dt = \boxed{\pi}$$



Example 2 Use Stokes' Theorem to evaluate \Rightarrow Find $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$

$\oint_C \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = \langle z^2, y^2, xy \rangle$ and C is the triangle defined by $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$.

S : plane, we need to find the equation using a point and the normal vector to the plane

We can get the normal vector by taking the cross product of two vectors in the plane.

Vector from $(1,0,0)$ to $(0,1,0)$

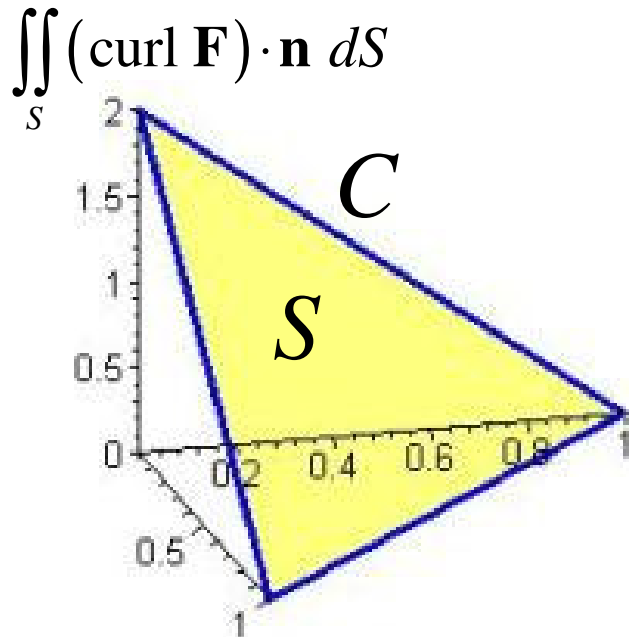
$$\mathbf{v}_1 = \langle 0-1, 1-0, 0-0 \rangle = \langle -1, 1, 0 \rangle$$

Vector from $(1,0,0)$ to $(0,0,2)$

$$\mathbf{v}_2 = \langle 0-1, 0-0, 2-0 \rangle = \langle -1, 0, 2 \rangle$$

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle 2, 2, 1 \rangle$$

For a plane we could have used \mathbf{v} and made it a unit vector to get \mathbf{n}



$S: 2x + 2y + z = d$ plug in any point to find d

$$(1,0,0) \rightarrow 2(1) + 2(0) + (0) = d = 2$$

$$\Rightarrow 2x + 2y + z = 2 \text{ so } \boxed{S: z = 2 - 2x - 2y}$$

$$g = 2x + 2y + z - 2 = 0$$

$$\nabla g = \langle 2, 2, 1 \rangle \quad \|\nabla g\| = \sqrt{4+4+1} = 3$$

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{1}{3} \langle 2, 2, 1 \rangle \quad dS = \sqrt{1 + (z_x)^2 + (z_y)^2} \, dA$$

$$dS = \sqrt{1 + (-2)^2 + (-2)^2} \, dA$$

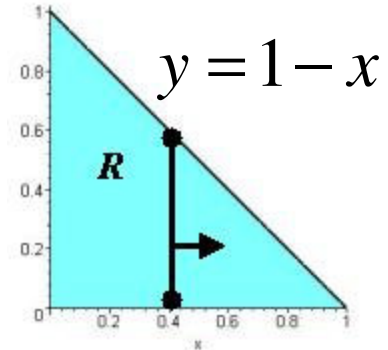
$$dS = 3 \, dA$$

Example 2 (cont.)

$$\mathbf{F} = \langle z^2, y^2, xy \rangle \quad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = \langle x, 2z - y, 0 \rangle$$

$$(\text{curl } \mathbf{F}) = \langle x, 2z - y, 0 \rangle \quad \mathbf{n} = \frac{1}{3} \langle 2, 2, 1 \rangle \quad dS = 3dA$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R (2x + 4z - 2y) \, dA$$



$$= \int_0^1 \int_0^{1-x} [2x + 4(2 - 2x - 2y) - 2y] \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} [8 - 6x - 10y] \, dy \, dx = \int_0^1 [(8 - 6x)y - 5y^2]_0^{1-x} \, dx$$

$$= \int_0^1 [(8 - 6x)(1 - x) - 5(1 - x)^2] \, dx = \int_0^1 (x^2 - 4x + 3) \, dx$$

$$= \left(\frac{x^3}{3} - 2x^2 + 3x \right)_0^1 = \frac{1}{3} - 2 + 3 = \boxed{\frac{4}{3}}$$