9.16 Divergence Theorem

- D: a closed and bounded region in 3-space
- S: the piecewise smooth boundary of D oriented outward
- $\mathbf{n}$: the unit normal to S, defines orientation of S
- $\mathbf{F}$: $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field

with P, Q, R, and all first partial derivatives continuous in a region of 3-space containing D

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\ dS = \iiint_D div\ \mathbf{F}\ dV$$

surface integral $\Leftrightarrow$ triple integral

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS = \iiint_D div\ \mathbf{F}\ dV$$

$\Rightarrow$

The majority of the time you will trade in the surface integral for the triple integral.

Use the divergence theorem to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS$ of the vector field $\mathbf{F} = 4x\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$ with D the region bounded by the sphere $x^2 + y^2 + z^2 = 4$.

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS = \iiint_D div\ \mathbf{F}\ dV$$

$$= \iiint_D (4+1+4)\ dV$$

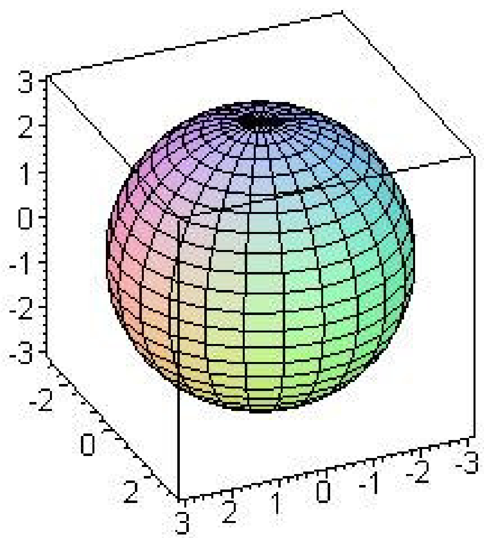
$$= 9\iiint_D dV$$

The volume of the solid

$$= 9\left[\frac{4}{3}\pi(2)^3\right]$$

$$= \boxed{96\pi}$$

Use the divergence theorem to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS$ of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ with D the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$.



$$0 \le \rho \le a$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS = \iiint_D div\ \mathbf{F}\ dV$$

$$= \iiint_D \left(3x^2 + 3y^2 + 3z^2\right)\ dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \left(3\rho^2\right)\rho^2 \sin\phi d\rho d\phi d\theta$$

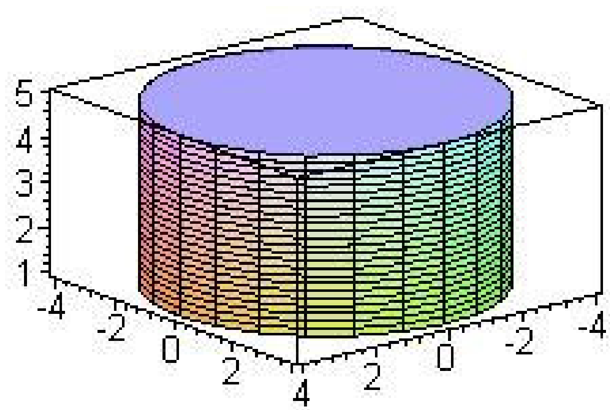
$$= \int_0^{2\pi}\int_0^{\pi} \left[\tfrac{3}{5}\rho^5\right]_0^a \sin\phi d\phi d\theta$$

$$= \tfrac{3}{5}a^5 \int_0^{2\pi}\int_0^{\pi} \sin\phi d\phi d\theta$$

$$= \tfrac{3}{5}a^5 \int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi} d\theta$$

$$= \tfrac{6}{5}a^5 \int_0^{2\pi} d\theta = \boxed{\frac{12\pi}{5}a^5}$$

Use the divergence theorem to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS$ of the vector field $\mathbf{F} = y^2\mathbf{i} + xz^3\mathbf{j} + (z-1)^2\mathbf{k}$ with D the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1, z = 5$



$$1 \le z \le 5$$
$$0 \le r \le 4$$
$$0 \le \theta \le 2\pi$$

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\ dS = \iiint_D div\ \mathbf{F}\ dV$$

$$div\ \mathbf{F} = 2(z-1)$$

$$= \int_0^{2\pi}\int_0^4\int_1^5 \left[2(z-1)\right]\ rdzdrd\theta$$

$$= 2\int_0^{2\pi}\int_0^4\int_1^5 \left[r(z-1)\right]\ dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^4 \left[r(z-1)^2\right]_1^5\ drd\theta$$

$$= 16\int_0^{2\pi}\int_0^4 rdrd\theta \quad = 16\left[\frac{r^2}{2}\right]_0^4 \int_0^{2\pi} d\theta$$

$$= 16\cdot 8\int_0^{2\pi} d\theta \quad = 16\cdot 8\cdot 2\pi = \boxed{256\pi}$$