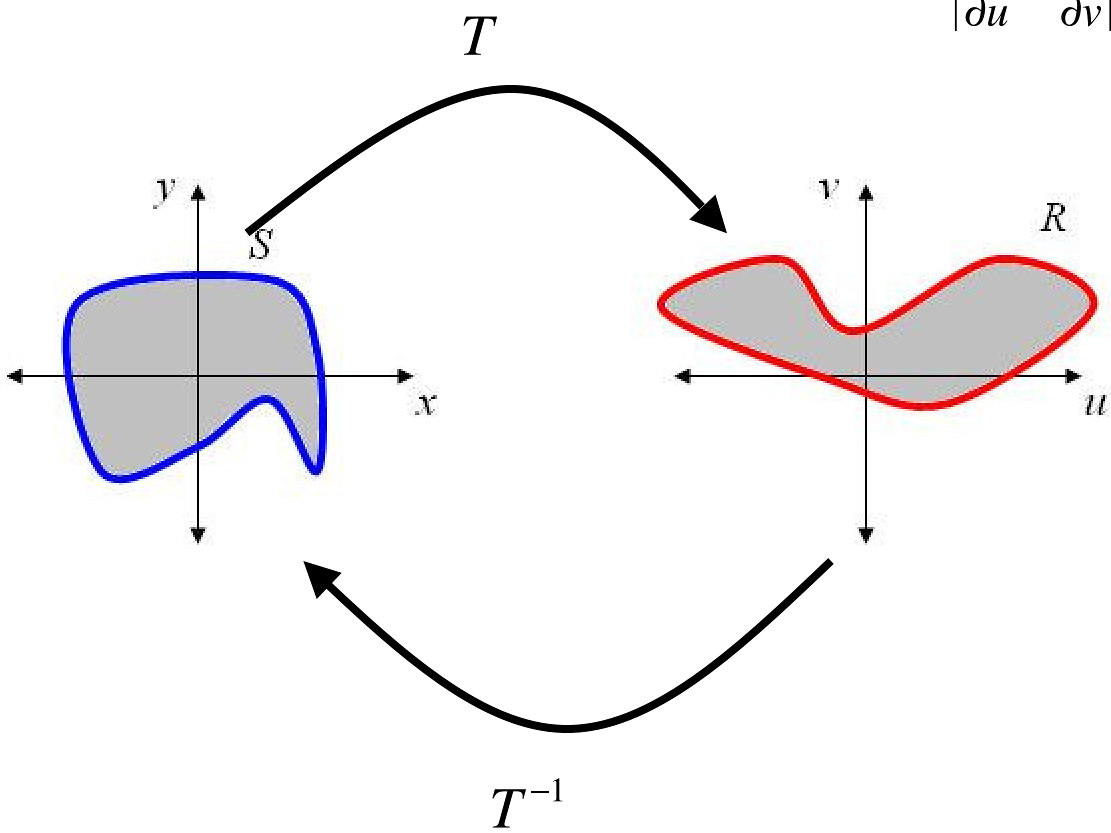
Change of variable

$$T : S \to R$$

$$(x, y) = \left(f(u,v), g(u,v)\right)$$

Jacobian

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$



element of area $dxdy$ becomes $\left|J(u,v)\right| dudv$

absolute value

Provided:

f and g have continuous first partial derivatives on S

T is one-to-one

R and S consist of a piecewise smooth simple closed curve and its interior

$J \neq 0$

$$\iint_S F(x, y)\, dA = \iint_R F\left(f(u,v), g(u,v)\right) \left|J(u,v)\right| dudv$$

Conversion of Rectangular into Polar:

$$x = r\cos\theta \qquad \frac{\partial x}{\partial r} = \cos\theta; \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$y = r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta; \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$dxdy = \left|J(r,\theta)\right| dr d\theta = \boxed{r dr d\theta}$$

or $dydx$

Conversion of Rectangular into Spherical:

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\frac{\partial x}{\partial \rho} = \sin\phi\cos\theta \qquad \frac{\partial x}{\partial \phi} = \rho\cos\phi\cos\theta \qquad \frac{\partial x}{\partial \theta} = -\rho\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial \rho} = \sin\phi\sin\theta \qquad \frac{\partial y}{\partial \phi} = \rho\cos\phi\sin\theta \qquad \frac{\partial y}{\partial \theta} = \rho\sin\phi\cos\theta$$

$$\frac{\partial z}{\partial \rho} = \cos\phi \qquad \frac{\partial z}{\partial \phi} = -\rho\sin\phi \qquad \frac{\partial z}{\partial \theta} = 0$$

$$J(\rho,\phi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi\left[\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right] - (-\rho\sin\phi)\left[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right]$$

$$= \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\sin^3\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) = \boxed{\rho^2\sin\phi}$$

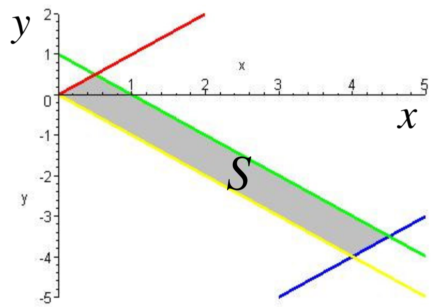
Evaluate

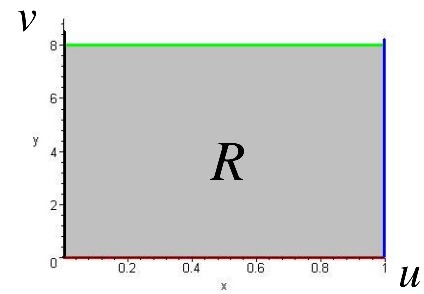
$$\iint_S e^{x+y}\sqrt[3]{x-y}\ dA$$

where S is the region bounded by the lines

$x+y=0;\ \ x+y=1$

$x-y=0;\ \ x-y=8$





Let

$u=x+y$

$v=x-y$

$0\le u\le 1$

$0\le v\le 8$

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$

2 choices:

a) Solve for x and y if possible

b) Use the inverse transformation

$$J(x,y)=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix} \text{ and } J(x,y)\cdot J(u,v)=1$$

$$\text{so } J(u,v)=\frac{1}{J(x,y)}$$

$\dfrac{\partial u}{\partial x}=1 \quad \dfrac{\partial u}{\partial y}=1$

$\dfrac{\partial v}{\partial x}=1 \quad \dfrac{\partial v}{\partial y}=-1$

$$J(x,y)=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-1-1=-2$$

$$J(u,v)=\frac{1}{J(x,y)} \qquad \Rightarrow J(u,v)=-\frac{1}{2}$$

$$\iint_S e^{x+y}\sqrt[3]{x-y}\ dA \ =\iint_R e^u\sqrt[3]{v}\left|J(u,v)\right|dudv$$

$$=\frac{1}{2}\int_0^8\int_0^1 e^u v^{1/3}dudv \quad =\frac{1}{2}(e-1)\int_0^8 v^{1/3}dudv$$

$$=\frac{1}{2}(e-1)\cdot\frac{3}{4}8^{4/3} \quad =\boxed{6(e-1)}$$

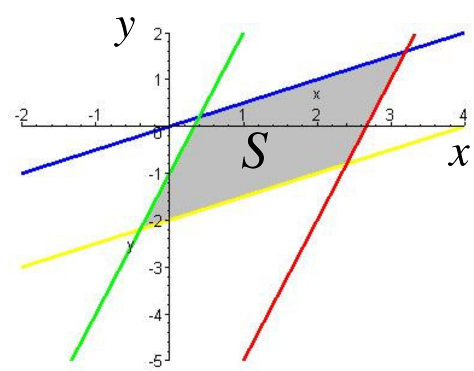
Evaluate

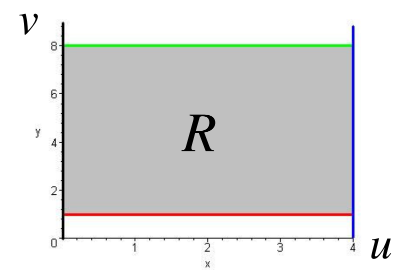
$$\iint_S \frac{x-2y}{3x-y}\,dA$$

where S is the region bounded by the lines

$x-2y=0;\quad x-2y=4$

$3x-y=1;\quad 3x-y=8$





Let

$u=x-2y$

$v=3x-y$

$$\frac{\partial u}{\partial x}=1 \qquad \frac{\partial u}{\partial y}=-2$$

$$\frac{\partial v}{\partial x}=3 \qquad \frac{\partial v}{\partial y}=-1$$

$$J(x,y)=\begin{vmatrix}1 & -2\\ 3 & -1\end{vmatrix}=-1+6=5$$

$$J(u,v)=\frac{1}{J(x,y)} \qquad \Rightarrow J(u,v)=\frac{1}{5}$$

$$\iint_S \frac{x-2y}{3x-y}\,dA=\iint_R \frac{u}{v}\left|J(u,v)\right|dudv$$

$$=\frac{1}{5}\int_1^8\int_0^4 \frac{u}{v}\,dudv \quad =\frac{1}{5}\cdot\left[\frac{u^2}{2}\right]_0^4\int_1^8 \frac{1}{v}\,dv$$

$$=\frac{8}{5}\int_1^8 \frac{1}{v}\,dv \quad =\boxed{\frac{8}{5}\ln 8}$$