

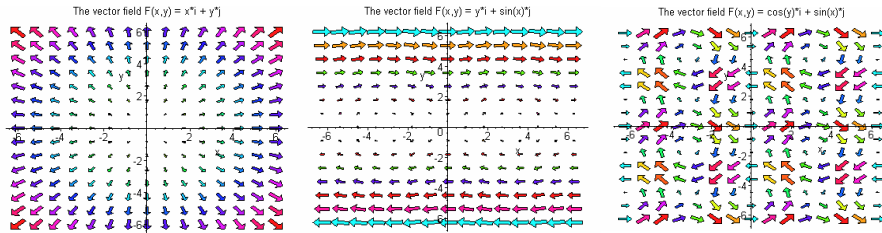
## Section 9.7 Divergence and Curl

"Del",  $\nabla$  - A defined operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

The **gradient** of a function (at a point) is a vector that points in the direction in which the function increases most rapidly.

A **vector field** is a vector function that can be thought of as a velocity field of a fluid. At each point it assigns a vector that represents the velocity of a particle at that point.

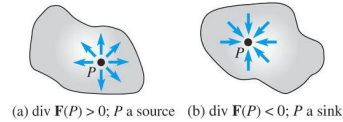


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The **flux** of a vector field is the volume of fluid flowing through an element of surface area per unit time.

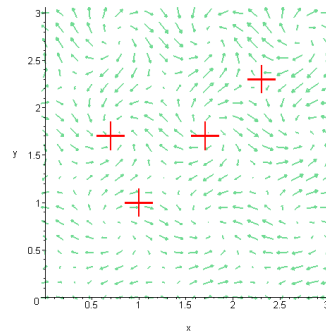
The **divergence** of a vector field is the flux per unit volume.

The divergence of a vector field is a number that can be thought of as a measure of the rate of change of the density of the fluid at a point.



The **curl** of a vector field measures the tendency of the vector field to rotate about a point.

The curl of a vector field at a point is a vector that points in the direction of the axis of rotation and has magnitude represents the speed of the rotation.



Vector Field

$$\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

Scalar Function

$$f(x, y, z)$$

Gradient  $grad(f)$ 

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Divergence  $div(\mathbf{F})$ 

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$$

Curl  $curl(\mathbf{F})$ 

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

$$\nabla \times \mathbf{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

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$$\mathbf{F}(x, y, z) = \langle xe^{-z}, 4yz^2, 3ye^{-z} \rangle$$

$$div(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xe^{-z}, 4yz^2, 3ye^{-z} \rangle = \boxed{e^{-z} + 4z^2 - 3ye^{-z}}$$

$$curl(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-z} & 4yz^2 & 3ye^{-z} \end{vmatrix} = \langle 3e^{-z} - 8yz^2, -(0 - (-xe^{-z})), 0 \rangle$$

$$\boxed{\langle 3e^{-z} - 8yz^2, -xe^{-z}, 0 \rangle}$$

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$grad(\text{scalar function}) = \text{Vector Field}$

$div(\text{Vector Field}) = \text{scalar function}$

$curl(\text{Vector Field}) = \text{Vector Field}$

Which of the 9 ways to combine grad, div and curl by taking one of each. Which of these combinations make sense?

<del><math>grad(grad(f))</math> Vector Field</del>	$\checkmark div(grad(f))$ Vector Field	$\checkmark curl(grad(f))$ Vector Field	<b>0</b> vector
$\checkmark grad(div(\mathbf{F}))$ scalar function	<del><math>div(div(\mathbf{F}))</math> scalar function</del>	<del><math>curl(div(\mathbf{F}))</math> scalar function</del>	
<del><math>grad(curl(\mathbf{F}))</math> Vector Field</del>	$\checkmark div(curl(\mathbf{F}))$ Vector Field	$\checkmark curl(curl(\mathbf{F}))$ Vector Field	<b>0</b> scalar

2 of the above are always zero.

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Verify the given identity. Assume continuity of all partial derivatives.

$$curl(grad(f)) = \mathbf{0}.$$

$$grad(f) = \langle f_x, f_y, f_z \rangle$$

$$curl(grad(f)) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\nabla \times \nabla f = \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle = \mathbf{0}$$

since mixed partial derivatives are equal.

Verify the given identity. Assume continuity of all partial derivatives.

$$div(curl(\mathbf{F})) = 0.$$

$$\text{Let } \mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \quad curl(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$div(curl(\mathbf{F})) = \nabla \cdot \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \begin{aligned} div(curl(\mathbf{F})) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\ div(curl(\mathbf{F})) &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\ div(curl(\mathbf{F})) &= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0 \end{aligned}$$

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