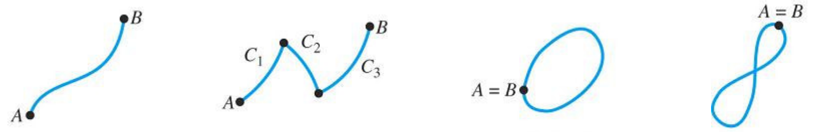
# Section 9.8 Line Integrals

Parametric Curve

$x = f(t), y = g(t)$



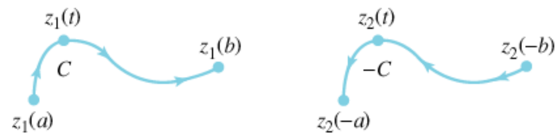

(a) Smooth curve — $f'$ and $g'$ continuous for $t$ in $[a,b]$

(b) Piecewise-smooth curve — Consists of a finite number of smooth curves

(d) Simple closed curve — Starts and ends at the same point and doesn't cross itself

(c) Closed but not simple — Starts and ends at the same pt.

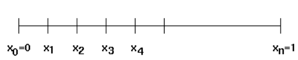
## Orientation of the curve

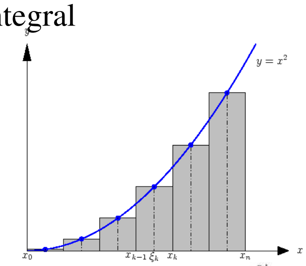



## Geometric Interpretation of a Line Integral

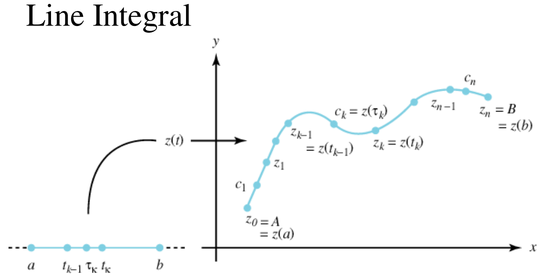
9.8

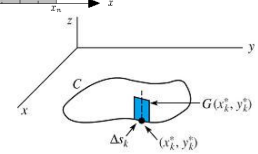
### Definite Integral



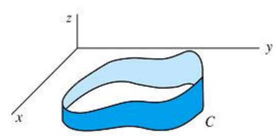




### Line Integral




(a) Vertical rectangle




(b) "Fence" or "curtain" of varying height $G(x, y)$ with base $C$

**9.8**

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \quad = \int_a^b U(t)\,f'(t)\,dt + V(t)\,g'(t)\,dt$$

Parametrize the path

$x = f(t),\ \ y = g(t)\ \ a \le t \le b$

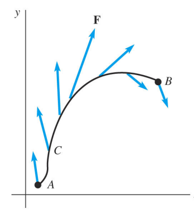
$dx = f'(t)\,dt,\ \ dy = g'(t)\,dt$

Substitute everything

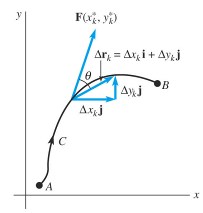
$P(x,y) = P\big(f(t), g(t)\big) = U(t)$

$Q(x,y) = Q\big(f(t), g(t)\big) = V(t)$

Work done by a vector field **F** as its point of application moves along $C$ from $A$ to $B$.



(a)



(b)

<u>Calc II</u>

$C \to line$

$W = \mathbf{F} \cdot d$

<u>Now</u>

$C \to curve$

$\mathbf{r}$ the parametrization of $C$ as a vector

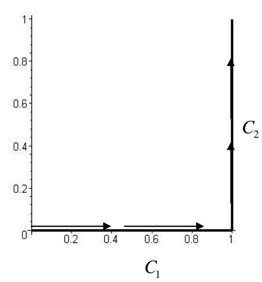
$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

## Section 9.8 #14

Evaluate $\int_C y\,dx + x\,dy$

on $C$ : line segments from $(0,0)$ to $(1,0)$

and from $(1,0)$ to $(1,1)$




$$\int_C y\,dx + x\,dy = \int_{C_1} y\,dx + x\,dy + \int_{C_2} y\,dx + x\,dy$$

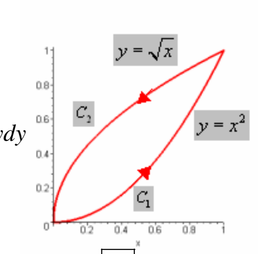
$$C_1 : \begin{matrix} x = t & y = 0 \\ dx = dt & dy = 0 \end{matrix} \quad 0 \le t \le 1 \qquad C_2 : \begin{matrix} x = 1 & y = t \\ dx = 0 & dy = dt \end{matrix} \quad 0 \le t \le 1$$

$$\int_{C_1} y\,dx + x\,dy = \int_0^1 (0+0)\,dt = 0 \qquad \int_{C_2} y\,dx + x\,dy = \int_0^1 (0+1)\,dt = 1$$

$$\int_C y\,dx + x\,dy = \boxed{1}$$

## Section 9.8 #20

Evaluate $\int_C \left(x^2+y^2\right)dx - 2xydy$ on the given closed curve $C$.




$$\int_C \left(x^2+y^2\right)dx-2xydy = \int_{C_1}\left(x^2+y^2\right)dx-2xydy+\int_{C_2}\left(x^2+y^2\right)dx-2xydy$$

$C_1$: $x=t \quad y=t^2$, $dx=dt \quad dy=2tdt$, $0\le t\le 1$

$$\int_{C_1}\left(x^2+y^2\right)dx-2xydy=\int_0^1\left(t^2+t^4-4t^4\right)dt=\int_0^1\left(t^2-3t^4\right)dt=\frac{t^3}{3}-\frac{3t^5}{5}\Big|_0^1=\frac{1}{3}-\frac{3}{5}=\frac{5-9}{15}=\boxed{\frac{-4}{15}}$$

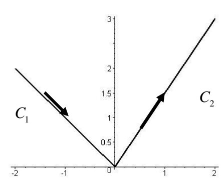
$C_2$: $x=t \quad y=\sqrt{t}$, $dx=dt \quad dy=\frac{1}{2\sqrt{t}}dt$, $t$ starts at 1 and ends at 0

$$\int_{C_2}\left(x^2+y^2\right)dx-2xydy=\int_1^0\left(t^2+t-t\right)dt=-\int_0^1 t^2dt=-\frac{t^3}{3}\Big|_0^1=\boxed{-\frac{1}{3}}$$

$$\int_C\left(x^2+y^2\right)dx-2xydy=\frac{-4}{15}-\frac{1}{3}=\frac{-4-5}{15}=\frac{-9}{15}=\boxed{\frac{-3}{5}}$$

## Section 9.8 #32

Find the work done by the force $\mathbf{F}(x,y)=2xy\mathbf{i}+4y^2\mathbf{j}$ acting along the piecewise smooth curve consisting of line segments from $(-2,2)$ to $(0,0)$ and from $(0,0)$ to $(2,3)$.




$$Work=\int_C \mathbf{F}\cdot dr=\int_{C_1}\mathbf{F}\cdot dr+\int_{C_2}\mathbf{F}\cdot dr$$

$C_1$: $x=t \quad y=-t$, $dx=dt \quad dy=-dt$, $-2\le t\le 0$

$r=t\mathbf{i}-t\mathbf{j} \Rightarrow dr=\langle 1,-1\rangle dt$

$\mathbf{F}$ on $C_1$: $\langle -2t^2, 4t^2\rangle$

$$\int_{C_1}\mathbf{F}\cdot dr=\int_{-2}^0\left(-2t^2-4t^2\right)dt$$

$$=\int_{-2}^0 -6t^2dt=-2t^3\Big|_{-2}^0=-2\left(0-(-8)\right)=\boxed{-16}$$

$C_2$: $x=t \quad y=\frac{3}{2}t$, $dx=dt \quad dy=\frac{3}{2}dt$, $0\le t\le 2$

$r=t\mathbf{i}+\frac{3}{2}t\mathbf{j} \Rightarrow dr=\left\langle 1,\frac{3}{2}\right\rangle dt$

$\mathbf{F}$ on $C_2$: $\langle 3t^2, 9t^2\rangle$

$$\int_{C_2}\mathbf{F}\cdot dr=\int_{C_2}\left(3t^2+\frac{27}{2}t^2\right)dt=\int_0^2\frac{33}{2}t^2dt$$

$$=\frac{11}{2}t^3\Big|_0^2=\frac{11}{2}(8-0)=\boxed{44}$$

$$Work=\int_C\mathbf{F}\cdot dr=-16+44=\boxed{28}$$