

Section 9.9 Independence of Path

$P(x, y)dx + Q(x, y)dy$ is called an **exact differential** if there exists a function $\phi(x, y)$ called a **potential function** such that $d\phi = P(x, y)dx + Q(x, y)dy$

$$\Rightarrow \frac{\partial \phi}{\partial x} = P(x, y) \text{ and } \frac{\partial \phi}{\partial y} = Q(x, y)$$

If this is the case, then the mixed partial of ϕ must be equal.

$$\Rightarrow \phi_{xy} = P_y \text{ should equal } \phi_{yx} = Q_x$$

$$\int_c Pdx + Qdy = \int_c \mathbf{F} \cdot d\mathbf{r}$$

When $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ and $\mathbf{r} = \langle x, y \rangle$

$$\Rightarrow \mathbf{F} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle = \nabla \phi$$

For 3 dimensions:

$$d\phi = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

$$\frac{\partial \phi}{\partial z} = R(x, y, z) \Rightarrow R_y = Q_z \text{ and } R_x = P_z$$

$$\mathbf{F} = \langle P, Q, R \rangle = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle = \nabla \phi$$

Since $\mathbf{F} = \nabla \phi$, then $\text{curl } \mathbf{F} = \text{curl}(\text{grad } \phi) = \mathbf{0}$

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A line integral whose value is the same for every curve connected the starting and ending point is called **independent of path**.

2-dimensions

$$\int_c Pdx + Qdy \text{ independent of path} \Leftrightarrow Pdx + Qdy \text{ is an exact differential}$$



$$P_y = Q_x$$

3-dimensions

$$\int_c Pdx + Qdy + Rdz \text{ independent of path} \Leftrightarrow Pdx + Qdy + Rdz \text{ is an exact differential}$$



$$\mathbf{F} = \langle P, Q, R \rangle$$

$$\text{curl } \mathbf{F} = \mathbf{0}$$

\mathbf{F} is called

conservative

Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a,b]$ such that $F(x)$ is a function whose derivative is $f(x)$ on $[a,b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Fundamental Theorem of Line Integrals

Let C be a curve with starting point (x_0, y_0) and ending point (x_1, y_1) .

Suppose $P(x, y)dx + Q(x, y)dy$ is an exact differential (there exists ϕ such that $d\phi = P(x, y)dx + Q(x, y)dy$).

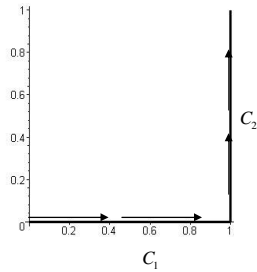
$$\int_C P(x, y)dx + Q(x, y)dy = \phi(x_1, y_1) - \phi(x_0, y_0)$$

$\int_C Pdx + Qdy$ independent of path $\left\{ \begin{array}{l} a) \text{ Find } \phi \text{ and evaluate it at the endpoints of } C \\ b) \text{ Pick a "convenient" path and evaluate the line integral. (convenient = horizontal or vertical)} \end{array} \right.$

Section 9.8 #14

Evaluate $\int_C ydx + xdy$

on C : line segments from $(0,0)$ to $(1,0)$ and from $(1,0)$ to $(1,1)$



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$$\int_C ydx + xdy = \int_{C_1} ydx + xdy + \int_{C_2} ydx + xdy$$

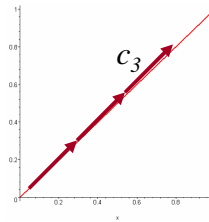
$$C_1: \begin{matrix} x=t & y=0 \\ dx=dt & dy=0 \end{matrix} \quad 0 \leq t \leq 1 \quad C_2: \begin{matrix} x=1 & y=t \\ dx=0 & dy=dt \end{matrix} \quad 0 \leq t \leq 1$$

$$\int_{C_1} ydx + xdy = \int_0^1 (0+0) dt = 0 \quad \int_{C_2} ydx + xdy = \int_0^1 (0+1) dt = 1 \quad \int_C ydx + xdy = \boxed{1}$$

Evaluate $\int_{C_3} ydx + xdy$

C_3 : line segment from $(0,0)$ to $(1,1)$

On C_3 : $\begin{matrix} x=t & y=t \\ dx=dt & dy=dt \end{matrix} \quad 0 \leq t \leq 1$



$$\int_{C_3} ydx + xdy = \int_0^1 tdt + tdt = \int_0^1 2tdt = \boxed{1}$$

Section 9.9 #12

Determine whether the given vector field is a gradient field.

If so, find a potential function ϕ for \mathbf{F} .

$$\mathbf{F} = 2xy^3\mathbf{i} + 3y^2(x^2 + 1)\mathbf{j}$$

$$P = 2xy^3 \quad Q = 3y^2(x^2 + 1)$$

For \mathbf{F} to be a gradient field, we need $P_y = Q_x$.

$$P_y = 6xy^2 \quad Q_x = 3y^2 \cdot 2x = 6xy^2, \text{ so } \mathbf{F} \text{ is a gradient field.}$$

$$\phi(x, y) = \int P dx = \int 2xy^3 dx = x^2 y^3 + G(y)$$

$$\phi(x, y) = \int Q dy = \int 3y^2(x^2 + 1) dy = y^3(x^2 + 1) + H(x)$$

$$\Rightarrow x^2 y^3 + G(y) = x^2 y^3 + y^3 + H(x)$$

$\Rightarrow H(x) = C_1$ since there is no function of x only on the LHS

$$\Rightarrow G(y) = y^3 + C_2$$

$$\boxed{\phi(x, y) = x^2 y^3 + y^3 + C}$$

Section 9.9 #18

Find the work done by the force

$\mathbf{F}(x, y) = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}$ along the indicated curve.

$$P = 2x + e^{-y} \quad Q = 4y - xe^{-y}$$

$$P_y = -e^{-y} \quad Q_x = -e^{-y} \Rightarrow \text{so } \mathbf{F} \text{ is a gradient field.}$$

$$\phi(x, y) = \int P dx = \int (2x + e^{-y}) dx = x^2 + xe^{-y} + G(y)$$

$$\phi(x, y) = \int Q dy = \int (4y - xe^{-y}) dy = 2y^2 + xe^{-y} + H(x)$$

$$\Rightarrow x^2 + xe^{-y} + G(y) = 2y^2 + xe^{-y} + H(x)$$

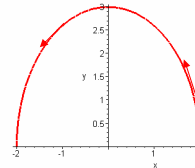
$$\Rightarrow H(x) = x^2 + C_1 \text{ and } G(y) = 2y^2 + C_2$$

$$\phi(x, y) = x^2 + xe^{-y} + 2y^2 + C$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(x, y) \Big|_{(2,0)}^{(-2,0)}$$

$$= (x^2 + xe^{-y} + 2y^2) \Big|_{(2,0)}^{(-2,0)}$$

$$= (4 - 2) - (4 + 2) = \boxed{-4}$$

**“Convenient” Path**

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (2x + e^{-y}) dx + (4y - xe^{-y}) dy$$

$$C: \quad x = t \quad y = 0$$

$$dx = dt \quad dy = 0$$

t starts at 2 and ends at -2

$$\int_C (2x + e^{-y}) dx + (4y - xe^{-y}) dy$$

$$= \int_2^{-2} (2t + 1) dt + 0 = - \int_{-2}^2 (2t + 1) dt$$

$$= -(t^2 + t) \Big|_{-2}^2 = -6 + 2 = \boxed{-4}$$

Section 9.9 #20

Show that the given integral is independent of path and evaluate.

$$\int_{(0,0,0)}^{(1,1,1)} 2x dx + 3y^2 dy + 4z^3 dz$$

$\mathbf{F} = \langle 2x, 3y^2, 4z^3 \rangle$ $\text{curl} \mathbf{F} = \mathbf{0} \Rightarrow$ so \mathbf{F} is a gradient field

$$\phi(x, y, z) = \int P dx = \int 2x dx = x^2 + G(y, z)$$

$$\phi(x, y, z) = \int Q dy = \int 3y^2 dy = y^3 + H(x, z)$$

$$\phi(x, y, z) = \int R dz = \int 4z^3 dz = z^4 + K(x, y)$$

$$\Rightarrow x^2 + G(y, z) = y^3 + H(x, z) = z^4 + K(x, y)$$

$$\Rightarrow H(x, z) = x^2 + z^4 + C_1, G(y, z) = y^3 + z^4 + C_2, \text{ and } K(x, y) = x^2 + y^3 + C_3$$

$$\phi(x, y, z) = x^2 + y^3 + z^3 + C \Rightarrow \int_{(0,0,0)}^{(1,1,1)} 2x dx + 3y^2 dy + 4z^3 dz = (x^2 + y^3 + z^3) \Big|_{(0,0,0)}^{(1,1,1)} = \boxed{3}$$

Section 9.9 #26

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{F} = \langle 2 - e^z, 2y - 1, 2 - xe^z \rangle$ $\mathbf{r} = \langle t, t^2, t^3 \rangle$ from $(-1, -1, -1)$ to $(2, 4, 8)$

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 - e^z & 2y - 1 & 2 - xe^z \end{vmatrix} = \langle 0, -e^z - (-e^z), 0 \rangle = \mathbf{0} \Rightarrow \text{so } \mathbf{F} \text{ is a gradient field}$$

$$\phi(x, y, z) = \int P dx = \int (2 - e^z) dx = 2x - xe^z + G(y, z)$$

$$\phi(x, y, z) = \int Q dy = \int (2y - 1) dy = y^2 - y + H(x, z)$$

$$\phi(x, y, z) = \int R dz = \int (2 - xe^z) dz = 2z - xe^z + K(x, y)$$

$$\Rightarrow 2x - xe^z + G(y, z) = y^2 - y + H(x, z) = 2z - xe^z + K(x, y)$$

$$G(y, z) = y^2 - y + 2z + C_1, H(x, z) = 2x - xe^z + 2z + C_2, \text{ and } K(x, y) = 2x + y^2 - y + C_3$$

$$\phi(x, y, z) = 2x - xe^z + y^2 - y + 2z + C$$

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = (2x - xe^z + y^2 - y + 2z) \Big|_{(-1,-1,-1)}^{(2,4,8)}$$

$$= (\mathcal{A} - 2e^8 + 16 - \mathcal{A} + 16) - (-2 + e^{-1} + \mathcal{A} - \mathcal{A} - 2) = \boxed{36 - 2e^8 - e^{-1}}$$