12.6 The Fourier-Bessel Series

Math 241 - Rimmer

$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$

parametric Bessel equation of order ν

has general solution on $(0, \infty)$ of

$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$

very important in the study of boundary-value problems involving partial differential equations expressed in cylindrical coordinates

$J_\nu(x)$ is called a **Bessel function of the first kind** of order ν.

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$Y_\nu(x)$ is called a **Bessel function of the second kind** of order ν.

For non-integer values of ν

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

For integer values (say n)

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

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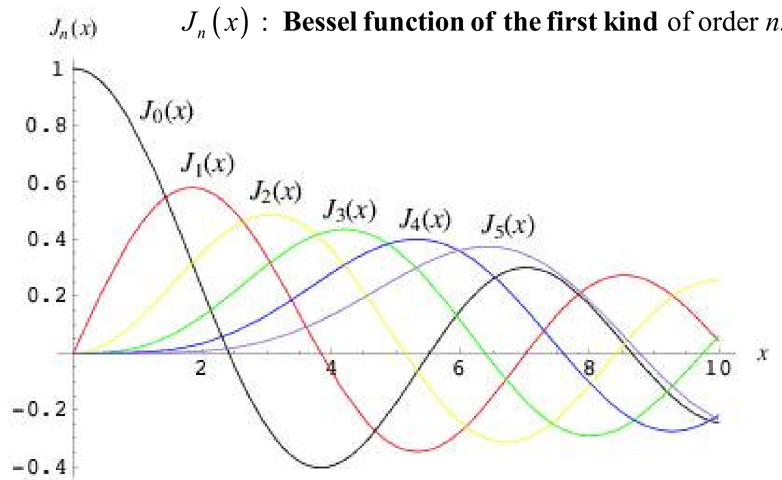
Let $\nu = n \quad n = 0, 1, 2, \ldots$

$J_n(x)$: **Bessel function of the first kind** of order n.



http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

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Let $\nu = n \quad n = 0,1,2,\ldots$

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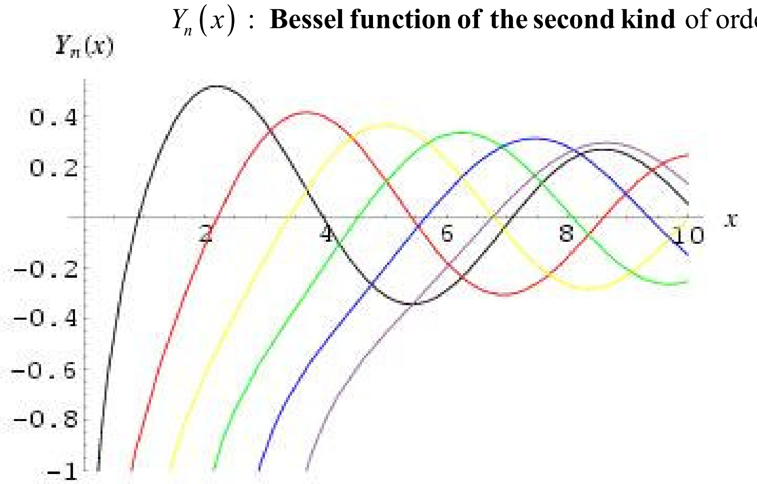

http://mathworld.wolfram.com/BesselFunctionoftheSecondKind.html

12.6 The Fourier-Bessel Series

Math 241 - Rimmer

Let $\nu = n \quad n = 0,1,2,\ldots$

The parametric Bessel differential equation becomes

$x^2 y'' + xy' + \left(\alpha^2 x^2 - n^2\right) y = 0$ with general solution on $(0,\infty)$ of $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$

The self-adjoint form of the differential equation is:

$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0$ we can identify $r(x) = x, q(x) = -\frac{n^2}{x}$, $p(x) = x$, and $\lambda = \alpha^2$

This is called a singular Sturm-Liouville problem when we add the following conditions:

$r(a) = 0$ and instead of 2 boundary conditions we only have $A_2 y(b) + B_2 y'(b) = 0$

For orthogonality, we need the solutions to be bounded at $x = a$.

$r(0) = 0$ and we will only need $A_2 y(b) + B_2 y'(b) = 0$

but $Y_n(0)$ is unbounded for all n, so for orthogonality we can only use $J_n(\alpha x)$

12.6 The Fourier-Bessel Series

The self-adjoint form of the differential equation is:

$$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0$$

we can identify $r(x) = x, q(x) = -\frac{n^2}{x},$

$p(x) = x,$ and $\lambda = \alpha^2$

with general solution

on $(0, \infty)$ of

$y = c_1 J_n(\alpha x)$

this gives a set of orthogonal functions

$\{J_n(\alpha_1 x), J_n(\alpha_2 x), \ldots J_n(\alpha_i x), \ldots\}$ $(\lambda_i = \alpha_i^2 \quad i = 1, 2, \ldots)$

that are orthogonal with respect to the weight function

$p(x) = x$ on the interval $[0, b]$

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x)\, dx = 0, \quad \lambda_i \neq \lambda_j$$

but this is all contingent upon the α_i being defined by a boundary condition at $x = b$ of the type

$A_2 J_n(\alpha b) + B_2 J_n'(\alpha b) = 0$

by the chain rule

$J_n'(\alpha x) = J_n'(\alpha x) \frac{d}{dx}(\alpha x) = \alpha J_n'(\alpha x)$

so the condition becomes:

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0$$

12.6 The Fourier-Bessel Series

We are interested in taking a function $f(x)$ and expanding it using Fourier eigenfunction expansion.

So now for $n = 0, 1, 2, \ldots,$ we have the Bessel functions of order n that will serve as our set of orthogonal functions used in the eigenfunction expansion of $f(x)$:

Let $n = 2$ for instance

$\{J_2(\alpha_1 x), J_2(\alpha_2 x), \ldots J_2(\alpha_i x), \ldots\}$ is a set of orthogonal **eigenfunctions** that are orthogonal with respect to the weight function $p(x) = x$ on the interval $[0, b]$

with corresponding **eigenvalues** $\lambda_i = \alpha_i^2 \quad i = 1, 2, \ldots$

The expansion of $f(x)$ with Bessel functions $\{J_n(\alpha_i x)\}$ $i = 1, 2, \ldots$ is called a **Fourier − Bessel series**.

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad \text{where } c_i = \frac{\int_0^b x J_n(\alpha_i x) f(x)\, dx}{\|J_n(\alpha_i x)\|^2}$$

In order to find the coefficients c_i, we need 3 properties of the Bessel J function:

1. $J_n(-x) = (-1)^n J_n(x)$
2. $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$
3. $\dfrac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Three different versions of the boundary condition at $x = b$ lead to three different types of solutions

1. $J_n(\alpha b) = 0$
2. $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$
3. $J_0'(\alpha b) = 0$

we'll have 3 different results for $\left\| J_n(\alpha_i x) \right\|^2$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$c_i = \frac{2}{b^2 \left[J_{n+1}(\alpha_i b)\right]^2} \int_0^b x J_n(\alpha_i x) f(x)\, dx$$

when the α_i defined by the boundary condition $J_n(\alpha b) = 0$

example:

#8 $f(x) = x^2, \quad 0 < x < 1$

$J_2(\alpha) = 0$

$b = 1, n = 2, f(x) = x^2$

$$c_i = \frac{2}{\left[J_3(\alpha_i)\right]^2} \int_0^1 x^3 J_2(\alpha_i x)\, dx$$

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$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$c_i = \frac{2\alpha_i^2}{\left(\alpha_i^2 b^2 - n^2 + h^2\right)\left[J_n(\alpha_i b)\right]^2}\int_0^b x J_n(\alpha_i x) f(x)\,dx$$

when the α_i defined by

the boundary condition

$hJ_n(\alpha b)+\alpha b J_n'(\alpha b)=0$

example:

$\#6\ f(x)=1,\quad 0<x<2$

$J_0(2\alpha)+\alpha J_0'(2\alpha)=0$

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$$f(x)=c_1+\sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \qquad c_1=\frac{2}{b^2}\int_0^b x f(x)\,dx$$

$$c_i = \frac{2}{b^2\left[J_0(\alpha_i b)\right]^2}\int_0^b x J_0(\alpha_i x) f(x)\,dx$$

when the α_i defined by

the boundary condition $J_0'(\alpha b)=0$

example:

$\#4\ f(x)=1,\quad 0<x<2$

$J_0'(2\alpha)=0$

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example:

#8 $f(x)=x^2, \quad 0<x<1$

$J_2(\alpha)=0$

$$c_i=\frac{2}{\left[J_3(\alpha_i)\right]^2}\int_0^1 x^3 J_2(\alpha_i x)\,dx$$

let $t=\alpha_i x$ $\quad x=0\Rightarrow t=0$, $\quad x=1\Rightarrow t=\alpha_i$

$dt=\alpha_i dx \Rightarrow dx=\frac{1}{\alpha_i}dt$

$x=\frac{t}{\alpha_i}\Rightarrow x^3=\frac{t^3}{\alpha_i^3}$

$$c_i=\frac{2}{\left[J_3(\alpha_i)\right]^2}\int_0^{\alpha_i}\frac{t^3}{\alpha_i^3}J_2(t)\frac{dt}{\alpha_i}$$

$$c_i=\frac{2}{\alpha_i^4\left[J_3(\alpha_i)\right]^2}\int_0^{\alpha_i}t^3 J_2(t)\,dt$$

$\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$

$\Rightarrow \frac{d}{dt}\left[t^3 J_3(t)\right]=t^3 J_2(t)$

$$c_i=\frac{2}{\alpha_i^4\left[J_3(\alpha_i)\right]^2}\int_0^{\alpha_i}\frac{d}{dt}\left[t^3 J_3(t)\right]dt$$

$$c_i=\frac{2}{\alpha_i^4\left[J_3(\alpha_i)\right]^2}\left[t^3 J_3(t)\right]_0^{\alpha_i}=\frac{2\left[\alpha_i^3 J_3(\alpha_i)\right]}{\alpha_i^4\left[J_3(\alpha_i)\right]^2}$$

$$c_i=\frac{2}{\alpha_i J_3(\alpha_i)}$$

$$f(x)=2\sum_{i=1}^{\infty}\frac{1}{\alpha_i J_3(\alpha_i)}J_2(\alpha_i x)$$