1. Solve using an appropriate Fourier Transform

$$ku_{xx} = u_t,\ x>0,\ t>0 \text{ with conditions } u_x(0,t)=0,\ \ u(x,0)=\begin{cases}1, & 0<x<2\\ 0, & x>2\end{cases}$$

$x>0$, and $u_x(0,t)=0$ given $\Rightarrow$ use Fourier Cosine Transform

$$F_C\{ku_{xx}\} = F_C\{u_t\}$$

$$k\left[-\alpha^2 F_C\{u(x,t)\} - \underbrace{u_x(0,t)}_{0}\right] = \frac{d}{dt}\left[F_C\{u(x,t)\}\right]$$

Let $F = F_C\{u(x,t)\}$

$$-k\alpha^2 F = \frac{dF}{dt}$$

$$\frac{dF}{F} = -k\alpha^2 dt$$

$$\int \frac{dF}{F} = \int -k\alpha^2 dt$$

$$\ln(F) = -k\alpha^2 t + C$$

$$e^{\ln(F)} = e^{-k\alpha^2 t + C}$$

$$F = Ae^{-k\alpha^2 t}$$

$$F_C\{u(x,t)\} = Ae^{-k\alpha^2 t}$$

$$F_C\{u(x,0)\} = A$$

$$F_C\{u(x,0)\} = A$$

$$F_C\{u(x,0)\} = \int_0^2 1\cdot\cos(\alpha x)\,dx$$

$$F_C\{u(x,0)\} = \left.\frac{\sin(\alpha x)}{\alpha}\right|_0^2 = \frac{\sin(2\alpha)}{\alpha}$$

$$A = \frac{\sin(2\alpha)}{\alpha}$$

$$F_C\{u(x,t)\} = \frac{\sin(2\alpha)}{\alpha}e^{-k\alpha^2 t}$$

$$u(x,t) = F_C^{-1}\left[F_C\{u(x,t)\}\right]$$

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(2\alpha)}{\alpha} e^{-k\alpha^2 t}\cos(\alpha x)\,d\alpha}$$

2. Evaluate

$$\int_C \frac{1}{z^3+4z}\,dz$$

$$C: |z-i| = \frac{3}{2}$$

$$z^3+4z = z\left(z^2+4\right) = z\left(z-2i\right)\left(z+2i\right)$$

$$\frac{1}{z^3+4z} = \frac{1}{z\left(z-2i\right)\left(z+2i\right)} = \frac{A}{z} + \frac{B}{z-2i} + \frac{C}{z+2i}$$

$$A: z = 0$$

$$A = \frac{1}{\cancel{z}\left(0-2i\right)\left(0+2i\right)} = \frac{1}{\left(-2i\right)\left(2i\right)} = \frac{1}{4}$$

$$B: z = 2i$$

$$B = \frac{1}{2i\,\cancel{\left(z-2i\right)}\left(2i+2i\right)} = \frac{1}{2i\cdot 4i} = \frac{-1}{8}$$

$$C: z = -2i$$

$$B = \frac{1}{-2i\left(-2i-2i\right)\cancel{\left(z+2i\right)}} = \frac{1}{-2i\cdot\left(-4i\right)} = \frac{-1}{8}$$

$$\int_C \frac{1}{z^3+4z}\,dz = \int_C \left[\frac{\frac{1}{4}}{z} + \frac{\frac{-1}{8}}{z-2i} + \frac{\frac{-1}{8}}{z+2i}\right]dz \quad C: |z-i| = \frac{3}{2}$$

$$= \int_C \underbrace{\left[\frac{\frac{1}{4}}{z} + \frac{\frac{-1}{8}}{z-2i}\right]}_{\text{singularity inside } C} dz + \int_C \underbrace{\frac{\frac{-1}{8}}{z+2i}}_{\text{singularity outside } C} dz$$

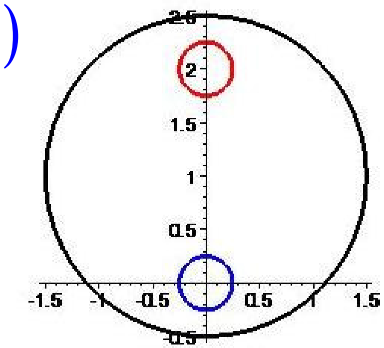
$$= \frac{1}{4}\int_{C_1} \frac{1}{z}\,dz - \frac{1}{8}\int_{C_2} \frac{1}{z-2i}\,dz + 0$$

$C_1: |z| = \frac{1}{4}$ and $C_2: |z-2i| = \frac{1}{4}$

$$= \frac{1}{4}\left(2\pi i\right) - \frac{1}{8}\left(2\pi i\right)$$

$$= \frac{1}{8}\left(2\pi i\right)$$

$$\boxed{= \frac{\pi i}{4}}$$

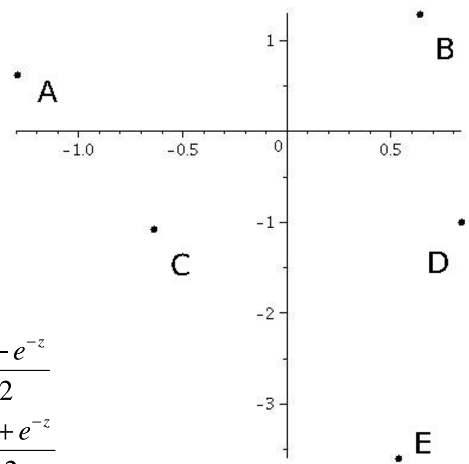


3. Match the following values to the plotted points.

1) $\sin(3+2i)$ 4) $\cosh(1-i)$

2) $\cos(2+i)$ 5) $\sin(i-1)$

3) $\sinh(1+i)$



1 radian $\approx 57^\circ$

2 radians $\approx 114^\circ$

3 radians $\approx 171^\circ$

$\cos 1 > 0$ $\sin 1 > 0$

$\cos 2 < 0$ $\sin 2 > 0$

$\cos 3 < 0$ $\sin 3 > 0$ but close to 0

Useful Formulas:

$\sin(z) = \sin x \cosh y + i \cos x \sinh y$

$\cos(z) = \cos x \cosh y - i \sin x \sinh y$

$\sinh(z) = \sinh x \cos y + i \cosh x \sin y$

$\cosh(z) = \cosh x \cos y + i \sinh x \sin y$

$\sinh(z) = \dfrac{e^z - e^{-z}}{2}$

$\cosh(z) = \dfrac{e^z + e^{-z}}{2}$

1) $\sin(3+2i) = \sin 3 \cosh 2 + i \cos 3 \sinh 2$ Re+ Im− **E**

2) $\cos(2+i) = \cos 2 \cosh 1 - i \sin 2 \sinh 1$ Re− Im− **C**

3) $\sinh(1+i) = \sinh 1 \cos 1 + i \cosh 1 \sin 1$ Re+ Im+ **B**

4) $\cosh(1-i) = \cosh 1 \cos(-1) + i \sinh 1 \sin(-1)$
$= \cosh 1 \cos 1 - i \sinh 1 \sin 1$ Re+ Im− **D**

5) $\sin(i-1) = \sin(-1)\cosh 1 + i \cos(-1) \sinh 1$
$= -\sin 1 \cosh 1 + i \cos 1 \sinh 1$ Re− Im+ **A**

The main difference between E and D is that E has a more negative imaginary part.

1) $\operatorname{Im}[\sin(3+2i)] = \underbrace{\cos 3}_{\text{close to } -1} \sinh 2$

4) $\operatorname{Im}[\cosh(1-i)] = \underbrace{-\sin 1}_{\text{close to } -1} \sinh 1$

So we are really comparing $\sinh 2$ to $\sinh 1$

$\sinh 2 > \sinh 1$

4. TRUE/FALSE. Decide on the truth value of the following statements.
An explanation is required in each case.

I) The set of points given by the equation $|z-1|\geq|z|$ can be described as $\operatorname{Re}(z)\leq\frac{1}{2}$.

II) If $|f(z)|\leq 2$ on the curve C given by $|z|=3$, then $\left|\oint_C f(z)\,dz\right|\leq 18\pi$.

III) $e^{\bar{z}}=\overline{e^{z}}$ for all z.

I) $|z-1|\geq|z| \Rightarrow$ Let $z=x+iy$

$$\sqrt{(x-1)^2+y^2}\geq\sqrt{x^2+y^2}$$

$$(x-1)^2+y^2\geq x^2+y^2$$

$$x^2-2x+1+y^2\geq x^2+y^2$$

$$-2x+1\geq 0$$

$$2x\leq 1$$

$$x\leq\frac{1}{2}\Rightarrow\operatorname{Re}(z)\leq\frac{1}{2}.$$

T

II) If $|f(z)|\leq 2$ on the curve C given by $|z|=3$, then $\left|\oint_C f(z)\,dz\right|\leq 18\pi$.

Bounding Theorem

f continuous on a smooth curve C and $|f(z)|\leq M$ for all z on C $\Rightarrow \left|\int_C f(z)\,dz\right|\leq ML$ ($L=$ length of C)

$$M=2, L=2\pi r=6\pi\Rightarrow\left|\oint_C f(z)\,dz\right|\leq 12\pi$$

T

III) Let $z=x+iy$, $e^{\bar{z}}=e^{x-iy}$

$$e^{x-iy}=e^x\left(\cos(-y)+i\sin(-y)\right)$$

$$=e^x\left(\cos(y)-i\sin(y)\right)$$

$$=e^x\cos(y)-ie^x\sin(y)=\overline{e^z}$$

T

5. Consider a semicircular plate of radius 1 whose boundary on the   is insulated $\left(u_\theta(r,0)=u_\theta(r,\pi)=0\right)$ and has its circular edge maintained at the temperature $u(1,\theta)=6\cos 2\theta-5\cos 3\theta$. The steady-state temperature $u(r,\theta)$ satisfies $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$.

Find $u\left(\frac{1}{2},\frac{\pi}{6}\right)$.

Assume : $u(r,\theta)=R(r)\Theta(\theta)$

$$\begin{cases} r^2R''+rR'-\lambda R=0 \\ \Theta''+\lambda\Theta=0 \end{cases}$$

$\underline{\boldsymbol{\lambda=0}}\Rightarrow \Theta''=0$

$\Theta(\theta)=c_1\theta+c_2$

$\Theta'(0)=c_1=0$

$\boxed{\Theta(\theta)=c_2}$

$\underline{\boldsymbol{\lambda<0}}$ say $\lambda=-\alpha^2\Rightarrow\Theta''-\alpha^2\Theta=0$

$\Theta(\theta)=c_3\cosh\theta+c_4\sinh\theta$

Hyperbolic Sine and Hyperbolic Cosine are not periodic. For this λ the only periodic solution is the trivial solution.

$\underline{\boldsymbol{\lambda>0}}$ say $\lambda=\alpha^2\Rightarrow\Theta''+\alpha^2\Theta=0$

$\Theta(\theta)=c_5\cos(\alpha\theta)+c_6\sin(\alpha\theta)$

$\Theta'(\theta)=-\alpha c_5\sin(\alpha\theta)+\alpha c_6\cos(\alpha\theta)$

$\Theta'(0)=\alpha c_6=0\Rightarrow c_6=0$

$\Theta(\theta)=c_5\cos(\alpha\theta)$

$\Theta'(\pi)=-\alpha c_5\sin(\alpha\pi)=0$

$\Rightarrow\alpha=n\quad ,n=1,2,3\ldots$

$\Rightarrow\lambda=n^2\quad ,n=1,2,3\ldots$

$\boxed{\Theta(\theta)=c_5\cos(n\theta)}$

#5 continued

$$r^2R'' + rR' - \lambda R = 0$$ This is a Cauchy-Euler differential equation.

$\underline{\boldsymbol{\lambda = 0}} \Rightarrow k^2 = 0$ Repeated root

$R(r) = c_7 + c_8 \ln r$

We run into a problem at $r = 0$,
since $\ln(r)$ is unbounded and our condition
of $u(r,\theta)$ bounded in the interior would be violated.

$\Rightarrow c_8 = 0$ and $\boxed{R(r) = c_7}$

$\underline{n = 0}$

$u_0(r,\theta) = R(r)\Theta(\theta) = c_7c_2 \Rightarrow \boxed{u_0 = A_0}$

$\underline{\boldsymbol{\lambda > 0}}$ we know $\lambda = n^2 \Rightarrow k^2 = n^2$ so $k = \pm n$

$R(r) = c_9r^n + c_{10}r^{-n}$

We run into a problem again at $r = 0$,
since $\frac{1}{\text{r}}$ is unbounded and our condition
of $u(r,\theta)$ bounded in the interior would be violated.

$\Rightarrow c_{10} = 0$ and $\boxed{R(r) = c_9r^n}$

$\underline{n = 1, 2, 3, \ldots}$

$u_n(r,\theta) = R(r)\Theta(\theta) = c_9r^n\left(c_5\cos(n\theta) + c_6\sin(n\theta)\right)$

$$\boxed{u_n(r,\theta) = r^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)}$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^nA_n\cos(n\theta)$$

$$u(1,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta)$$
$$= 6\cos 2\theta - 5\cos 3\theta$$

$\Rightarrow A_2 = 6, A_3 = -5,$ and $A_n = 0$ for $n \neq 2$ or 3

$$u(r,\theta) = 6r^2\cos 2\theta - 5r^3\cos 3\theta$$

$$u\left(\frac{1}{2},\frac{\pi}{6}\right) = 6\left(\frac{1}{4}\right)\cos\left(\frac{\pi}{3}\right) - 5\left(\frac{1}{8}\right)\cos\left(\frac{\pi}{2}\right)$$

$$\boxed{u\left(\frac{1}{2},\frac{\pi}{6}\right) = \frac{3}{4}}$$

6. Evaluate the following contour integrals, express the answer in $a+bi$ form.

$a)\ \oint_{C_1} |z-1|^2\, dz$ where C_1 is the unit circle

$b)\ \int_{C_2} e^z dz$ where C_2 is the straight line segment joining 1 to $1+\pi i$

$c)\ \int_{\frac{1}{2}}^{\frac{\pi}{4}i} z e^{2z} dz$

$a)\ \oint_{C_1} |z-1|^2\, dz$ where C_1 is the unit circle

$z=\cos t + i\sin t \qquad dz = (-\sin t + i\cos t)\,dt$

$z-1 = (\cos t - 1) + i\sin t$

$|z-1|^2 = (\cos t-1)^2 + \sin^2 t$

$|z-1|^2 = \cos^2 t - 2\cos t + 1 + \sin^2 t$

$|z-1|^2 = 2(1-\cos t)$

$|z-1|^2\, dz = 2(1-\cos t)(-\sin t + i\cos t)\,dt$

$|z-1|^2\, dz = 2(-\sin t + \sin t\cos t + i\cos t - i\cos^2 t)\,dt$

$\oint_{C_1} |z-1|^2\, dz = 2\int_0^{2\pi} (-\sin t + \sin t\cos t + i\cos t - i\cos^2 t)\,dt$

$= -2i\int_0^{2\pi} \cos^2 t\, dt$

$= -2i\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt$

$= -i\int_0^{2\pi} (1-\cos 2t)\,dt$

$= \boxed{-2\pi i}$

$b)$ The Fundamental Theorem of Contour Integrals $\Rightarrow$

$\int_{C_2} e^z dz$ where C_2 is the straight line segment joining 1 to $1+\pi i$

$= e^z\Big|_1^{1+\pi i} = e^{1+\pi i} - e^1 = e\left(e^{\pi i} - 1\right) = \boxed{-2e}$

$c)\ \int_{\frac{1}{2}}^{\frac{\pi}{4}i} z e^{2z} dz = \left[\tfrac{1}{2} z e^{2z} - \tfrac{1}{4} e^{2z}\right]_{\frac{1}{2}}^{\frac{\pi}{4}i} = \left[\tfrac{\pi}{8} i e^{\frac{\pi}{2}i} - \tfrac{1}{4} e^{\frac{\pi}{2}i}\right] - \left[\tfrac{1}{4}e - \tfrac{1}{4}e\right]$

$= \tfrac{1}{4} e^{\frac{\pi}{2}i}\left[\tfrac{\pi}{2} i - 1\right]$

$= \tfrac{1}{4} i\left[\tfrac{\pi}{2} i - 1\right]$

$= \boxed{-\tfrac{\pi}{8} - \tfrac{1}{4} i}$

$\underline{D}$	$\underline{I}$
z	e^{2z}
1	$\tfrac{1}{2}e^{2z}$
0	$\tfrac{1}{4}e^{2z}$

7. Answer the following questions.

a) Find all solutions to the equation $z^5 = 3\sqrt{3} - 3i$

b) Find the principle value of $\left(1+\sqrt{3}i\right)^{i/2}$ in $a+bi$ form.

a) Find all solutions to the equation $z^5 = 3\sqrt{3} - 3i$.

$$z = \left(3\sqrt{3} - 3i\right)^{1/5}$$

$$r = 6, \theta = -\frac{\pi}{6} \qquad n = 5$$

The n^{th} roots of z are

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \quad k = 0,1,2,\ldots n-1$$

$$k=0: \frac{\frac{-\pi}{6}}{5} = \frac{-\pi}{30} \Rightarrow w_0 = 6^{1/5}\left[\cos\left(\frac{-\pi}{30}\right) + i\sin\left(\frac{-\pi}{30}\right)\right]$$

$$k=1: \frac{\frac{-\pi}{6} + 2\pi}{5} = \frac{11\pi}{30} \Rightarrow w_1 = 6^{1/5}\left[\cos\left(\frac{11\pi}{30}\right) + i\sin\left(\frac{11\pi}{30}\right)\right]$$

$$k=2: \frac{\frac{-\pi}{6} + 4\pi}{5} = \frac{23\pi}{30} \Rightarrow w_2 = 6^{1/5}\left[\cos\left(\frac{23\pi}{30}\right) + i\sin\left(\frac{23\pi}{30}\right)\right]$$

$$k=3: \frac{\frac{-\pi}{6} + 6\pi}{5} = \frac{35\pi}{30} \Rightarrow w_3 = 6^{1/5}\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right]$$

$$k=4: \frac{\frac{-\pi}{6} + 8\pi}{5} = \frac{47\pi}{30} \Rightarrow w_4 = 6^{1/5}\left[\cos\left(\frac{47\pi}{30}\right) + i\sin\left(\frac{47\pi}{30}\right)\right]$$

b) Find the principle value of $\left(1+\sqrt{3}i\right)^{i/2}$ in $a+bi$ form.

$$\left(1+\sqrt{3}i\right)^{i/2} = e^{\frac{i}{2}\mathrm{Ln}\left(1+\sqrt{3}i\right)}$$

$$\mathrm{Ln}\left(1+\sqrt{3}i\right) = \ln 2 + \frac{\pi}{3}i$$

$$\frac{i}{2}\mathrm{Ln}\left(1+\sqrt{3}i\right) = \frac{i}{2}\left[\ln 2 + \frac{\pi}{3}i\right]$$

$$\frac{i}{2}\mathrm{Ln}\left(1+\sqrt{3}i\right) = -\frac{\pi}{6} + i\ln\sqrt{2}$$

$$e^{\frac{i}{2}\mathrm{Ln}\left(1+\sqrt{3}i\right)} = e^{-\frac{\pi}{6} + i\ln\sqrt{2}} = e^{-\frac{\pi}{6}}e^{i\ln\sqrt{2}}$$

$$\left(1+\sqrt{3}i\right)^{i/2} = \boxed{e^{-\frac{\pi}{6}}\cos\left(\ln\sqrt{2}\right) + ie^{-\frac{\pi}{6}}\sin\left(\ln\sqrt{2}\right)}$$

8. Answer the following questions.

$a)$ Show $f(z)=\dfrac{z^2}{|z|^2}$ does not have a limit as $z\to 0$.

$b)$ Evaluate $\displaystyle\lim_{z\to 1+i}\frac{z^4+4}{z^2-2z+2}$.

$c)$ Find the conjugate harmonic function $v(x,y)$ to the function $u(x,y)=xy^3-x^3y$.

$a)$ Show $f(z)=\dfrac{z^2}{|z|^2}$ does not have a limit as $z\to 0$.

Let $z=x+iy\Rightarrow \dfrac{z^2}{|z|^2}=\dfrac{x^2-y^2+2xyi}{x^2+y^2}$

$$\lim_{z\to 0}\frac{z^2}{|z|^2}\text{ along } x-axis=\frac{x^2}{x^2}=1$$

$$\lim_{z\to 0}\frac{z^2}{|z|^2}\text{ along } y-axis=\frac{-y^2}{y^2}=-1$$

Different limits $\Rightarrow \displaystyle\lim_{z\to 0}\frac{z^2}{|z|^2}$ does not exist

$b)$ Evaluate $\displaystyle\lim_{z\to 1+i}\frac{z^4+4}{z^2-2z+2}$

$$(1+i)^2=2i$$

$$(1+i)^4=2i\cdot 2i=-4$$

$$\frac{(1+i)^4+4}{(1+i)^2-2(1+i)+2}=\text{"}\frac{0}{0}\text{"}$$

$$\lim_{z\to 1+i}\frac{z^4+4}{z^2-2z+2}\overset{L'H}{=}\lim_{z\to 1+i}\frac{4z^3}{2z-2}$$

$$(1+i)^3=2i(1+i)=2i-2$$

$$2(1+i)-2=2[1+i-1]=2i$$

$$\lim_{z\to 1+i}\frac{4z^3}{2z-2}=\frac{4(2i-2)}{2i}=4\left(1-\frac{1}{i}\right)$$

$$=\boxed{4(1+i)}$$

$c)$ Find the conjugate harmonic function $v(x,y)$ to the function $u(x,y)=xy^3-x^3y$.

$$u(x,y)=xy^3-x^3y$$

$$u_x=y^3-3x^2y\overset{C-R}{=}v_y \qquad u_y=3xy^2-x^3\overset{C-R}{=}-v_x$$

$$v=\int v_y\,dy=\int\left(y^3-3x^2y\right)dy$$

$$v=\frac{y^4}{4}-\frac{-3x^2y^2}{2}+K(x)$$

$$v_x=-3xy^2+K'(x)$$

$$u_y=-v_x\Rightarrow 3xy^2-x^3=3xy^2-K'(x)$$

$$\Rightarrow K'(x)=x^3$$

$$\Rightarrow K(x)=\frac{x^4}{4}+C$$

$$\boxed{v(x,y)=\frac{y^4}{4}-\frac{3x^2y^2}{2}+\frac{x^4}{4}+C}$$