Homework for fouth test to be given Thursday Dec. 4

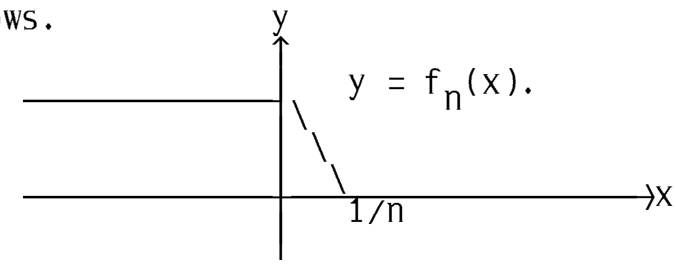
Pointwise convergence. If f_n is a sequence of real valued functions defined on a closed interval $[a,b]$ we say the sequence f_n converges pointwise to the function f on $[a,b]$ if for each $x \in [a,b]$ we have $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$.

Note if f_n is a sequence of continuous functions and f_n converges to f pointwise then the limiting function f need not be continuous. Suppose the functions f_n are defined as follows.

$f_n(x) = 1$ for $x \leqq 0$

$f_n(x) = 1 - nx$ for $0 < x < 1/n$

$f_n(x) = 0$ for $x \geqq 0$



Let $f(x) = 1$ for $x \leqq 0$ and $f(x) = 0$ for $x > 0$.
Prove $f_n(x)$ converges pointwise to f on $[-1,1]$.
Using the defintion of continuity prove f is not continuous at $x = 0$.
(i.e. find an explicit $\epsilon > 0$ so there is no $\delta > 0$ so that.....)

Uniform convergence. A sequence f_n of continuous functions defined on a closed interval $[a,b]$ are said to converge uniformly to a function f on $[a,b]$ if for every $\epsilon > 0$ there is an integer N so that if $n \geqq N$ and $x \in [a,b]$ then $|f_n(x) - f(x)| < \epsilon$.

Note for pointwise convergence the integer N depends on ϵ and x (i.e. $N = N(\epsilon,x)$) and for uniform convergence the integer N depends only on ϵ.

Prove that if f_n is a sequence of real valued continuous functions on the closed interval $[a,b]$ and f_n converges to f uniformly on $[a,b]$ then the limiting functions f is continuous. (You are proving the uniform limit of continuous functions is a continuous function.)

Prove $\sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{1}{n-1}$ Hint. Note $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

Let $f_n(x) = \sum_{k=1}^{n} k^{-2}\sin(kx)$ Prove f_n coverges to a function f as $n \longrightarrow \infty$.
You may use the fact that $|\sin(x)| \leqq 1$ for all x.

Prove the limiting function f is continuous.
Is the limiting function uniformly continous on $(-\infty,\infty)$? Note $f(x) = f(x+2\pi)$