

Homework for third test to be given Thursday Dec. 4

Taylor's Theorem with Lagrange form of the remainder. Suppose f is $(n+1)$ times differentiable on an open interval containing x and a . Then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x,a)$$

where $R_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$ where c lies between x and a .

Definitions . A set S of real numbers is open if for each $x \in S$ there is an $\epsilon > 0$ so that if $|x - y| < \epsilon$ then $y \in S$. A set is closed if its complement is open. If S is a set the closure of S is the intersection of all closed sets containing S . If S is a set then x is an accumulation point of S if for each $\epsilon > 0$ there is a $y \in S$ with $y \neq x$ and $|x - y| < \epsilon$. We define the null set to be both open and closed as is the whole real line $(-\infty, \infty)$.

Theorem. Suppose $\{S_\alpha\}$ is a collection of open sets and S is the union of the sets in the collection (i.e. $S = \bigcup_\alpha S_\alpha$) then S is open. If $S_1, S_2, S_3, S_4, \dots, S_n$ are a finite collection of open sets. If S is the union of all the sets (i.e. $S = S_1 \cup S_2 \cup \dots \cup S_n$) then S is open.

Theorem. Suppose $\{S_\alpha\}$ is a collection of closed sets and S is the intersection of the sets in the collection (i.e. $S = \bigcap_\alpha S_\alpha$) then S is closed. If S_1, S_2, \dots, S_n are a finite collection of closed sets. Then if S is the intersection of all the sets (i.e. $S = S_1 \cap S_2 \cap \dots \cap S_n$) is closed.

Definition. If S is a set of real numbers then x is an accumulation point of S if for every $\epsilon > 0$ there is a $y \in S$ with $y \neq x$ so that $|x - y| < \epsilon$.

Note a set S is closed if and only if it contains its accumulation point. If S is a set and Y is the set of accumulation points then the union of S and Y (i.e. $Y \cup S$) is the closure of S .

Definition. If S is a set we say $\{O_\alpha\}$ is an open cover of S if each of the sets O_α is open and the union of the O_α contains S . We say the open cover $\{O_\alpha\}$ has a finite subcover if there is a finite number of sets $O_{\alpha_1}, O_{\alpha_2}, \dots, O_{\alpha_n}$ which cover S .

Definition. A set S is compact if and only if every open cover of S has a finite subcover.

Theorem. (Heine-Borel) If S is a set of real numbers then S is compact if and only if S is closed and bounded.

Definition. A real valued function f defined on a domain D is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x, y \in D$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Theorem. If f is defined and continuous on the closed interval $[a,b]$ then f is uniformly continuous.

1. The function e^x is differentiable for all x and $\frac{d}{dx} e^x = e^x$ and it is known that $e^x > 0$ for all x . Suppose f is a function which is differentiable in then an open interval (a,b) and $f'(x) = kf(x)$ where k is a constant. Prove $f(x) = Ce^{kx}$ where C is a constant. (Hint let $g(x) = f(x)e^{-kx}$ and use the mean value theorem.)
2. Prove $e^x e^y = e^{x+y}$. (Hint use 1.)
3. Prove $e^x = 1 + x + \frac{x^2}{2!} + \dots$ where the series converges for all x . (i.e. prove the remainder $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$) Use first theorem of notes.
4. Prove $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ (i.s. prove the remainder $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$).
5. Prove that a set is closed if and only if it contains its accumulation points.
6. Suppose f is a function defined on \mathbb{R} . Show f is continuous if and only if the inverse image of every open set is open.
7. For the following sets determine if they are open, closed or neither. If they are not closed determine their closure.
 - A. All $x \in (0,1)$ except $x = 1/n$ for $n = 1,2,\dots$
 - B. All $x \in [0,1]$ which can be expressed as $x = n2^{-m}$ with n and m integers.
 - C. The positive integers. $n = 1,2,\dots$
 - D. All $x \in [0,1]$ which can be expressed in decimal without a 7.
e.g. $.7$ can be so expressed since $.7 = .699999\dots$ but $.71$ can not.
 - E. All $x \in (0,1)$ except for points x_n for $n = 1,2,\dots$ and $x_n \rightarrow 1$ as $n \rightarrow \infty$.
8. Suppose S is a set with the following property. If f is a bounded continuous function defined on the whole real line and $a = \sup\{f(x) \mid x \in S\}$ then there is a point $x_0 \in S$ so that $f(x_0) = a$. Prove that S is a closed and bounded set. Show that if S is not closed and bounded there is a bounded continuous function f so that if $a = \inf\{f(x) \mid x \in S\}$ then $f(x) < a$ for all $x \in S$.
9. Prove that if f is defined and differentiable on the whole real line and $f'(x)$ is bounded then f is uniformly continuous. (Hint, use the mean value theorem.)
10. Prove that if f is defined and continuous on $[0,\infty)$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$ then f is uniformly continuous on $[0,\infty)$. (You may use the fact that a function which is continuous on a closed interval $[a,b]$ is uniformly continuous on $[a,b]$).