

Homework for third test to be given Tuesday Dec. 4

1. Show that if f is strictly increasing (so $f(x) > f(y)$ for $x > y$) then $\lim_{x \rightarrow 0^+} f(x) = b$ exists and $b \geq f(0)$.
2. Suppose $f(x) = 0$ for x irrational and $f(x) = 1/n$ for $x = m/n$ in lowest terms. Show f is discontinuous at the rationals and continuous at the irrationals.

Mean Value Theorem. Suppose f is defined and continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b) . Then there is a $c \in (a,b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

3. Prove that if f is differentiable on the open interval (a,b) and $f'(x) = 0$ for $x \in (a,b)$ then $f(x) = C$ (a constant).
4. Given the function $f(x) = e^x$ is differentiable and $f'(x) = f(x)$ for $x \in \mathbb{R}$. Prove that if g is differentiable on the whole real line and $g'(x) = kg(x)$ (with k constant) then $g(x) = Ce^{kx}$ for $x \in \mathbb{R}$. (Hint, consider $h(x) = g(x)/f(kx)$).
5. Suppose f is differentiable for $x \in \mathbb{R}$ and $f(x+y) = f(x)f(y)$ for $x,y \in \mathbb{R}$. Prove $f(x) = Ce^{kx}$ for $x \in \mathbb{R}$ where k and C are constants.
6. Prove that if f is differentiable in the open interval (a,b) and $f'(x) > 0$ for $x \in (a,b)$ then f is strictly increasing.
7. Show that if f is defined and continuous in the closed interval $[a,b]$ and f is twice differentiable in the open interval (a,b) and $f''(x) > 0$ for $x \in (a,b)$ then if $c \in (a,b)$ the point $(c,f(c))$ lies below the straight line through $(a,f(a))$ and $(b,f(b))$. (This means f is strictly convex.)
8. Suppose h is differentiable for $x > 0$ and $h'(x) = 1/x$ and $h(1) = 0$. Prove that $h(xy) = h(x) + h(y)$ for $x,y > 0$.
9. Prove $h(x/y) = h(x) - h(y)$ for $x,y > 0$.
10. Prove $\frac{1}{2} < h(2) < 1$. Prove $h(e) = 1$ where $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.
11. Prove that $h(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $h(x) \rightarrow -\infty$ as $x \rightarrow 0^+$. (Hint $h(2^n) = nh(2)$)
12. Prove there is a unique continuous function e^x defined on the whole real line so that $e(h(x)) = x$. Show $e(x+y) = e(x)e(y)$ for $x,y \in \mathbb{R}$. Show $e'(x) = e(x)$ (You may use the theorem that if f is the inverse function to g so $f(g(x)) = x$ and g and f is differentiable then $f'(g(x)) = 1/g'(x)$).

Taylor's Theorem with Lagrange form of the remainder. Suppose f is $(n+1)$ times differentiable on an open interval containing x and a . Then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x,a)$$

$$\text{where } R_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \text{ where } c \text{ lies between } x \text{ and } a.$$

13. Prove $e^x = 1 + x + \frac{x^2}{2!} + \dots$ where the series converges for all x . (i.e. prove the remainder $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$)

Definitions. A set S of real numbers is open if for each $x \in S$ there is an $\epsilon > 0$ so that if $|x - y| < \epsilon$ then $y \in S$. A set is closed if its complement is open. If S is a set the closure of S is the intersection of all closed set containing S . If S is a set then x is an accumulation point of S if for each $\epsilon > 0$ there is a $y \in S$ with $y \neq x$ and $|x - y| < \epsilon$.

14. Prove that a set is closed if and only if it contains its accumulation points.
15. Suppose f is a function defined on \mathbb{R} . Show f is continuous if and only if the inverse image of every open set is open.
16. For the following sets determine if they are open, closed or neither. If they are not closed determine their closure.
 - A. All $x \in (0,1)$ except $x = 1/n$ for $n = 1,2,\dots$
 - B. All $x \in [0,1]$ which can be expressed as $x = n2^{-m}$ with n and m integers.
 - C. The positive integers. $n = 1,2,\dots$
 - D. All $x \in [0,1]$ which can be expressed in decimal without a 7.
e.g. $.7$ can be so expressed since $.7 = .699999\dots$ but $.71$ can not.
 - E. All $x \in (0,1)$ except for points x_n for $n = 1,2,\dots$ and $x_n \rightarrow 1$ as $n \rightarrow \infty$.
17. Suppose f is defined for all real x and $|f(x) - f(y)| \leq 5(x - y)$ for all real x and y .
 - a) Prove f is continuous.
 - b) Prove f is differentiable with derivative $f'(x) = 0$.
 - c) Prove f is constant $f(x) = C$ for all x .
18. Suppose S is a set with the following property. If f is a bounded continuous function defined on the whole real line and $a = \sup\{f(x) \mid x \in S\}$ then there is a point $x_0 \in S$ so that $f(x_0) = a$. Prove that S is a closed and bounded set. Show that if S is not closed and bounded there is a bounded continuous function f so that if $a = \inf\{f(x) \mid x \in S\}$ then $f(x) < a$ for all $x \in S$.
- Definition.** A real valued function f defined on a domain D is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x, y \in D$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
19. Prove that if f is defined and differentiable on the whole real line and $f'(x)$ is bounded then f is uniformly continuous. (Hint, use the mean value theorem.)
20. Prove that if f is defined and continuous on $[0, \infty)$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$ then f is uniformly continuous on $[0, \infty)$. (You may use the fact that a function which is continuous on a closed interval $[a, b]$ is uniformly continuous on $[a, b]$).