Homework.2 (Due Oct. 4) Answers

1. Suppose $\{x_n\}$ is a Cauchy sequence in a normed linear space. Suppose $\{y_n\}$ is a subsequence of $\{x_n\}$. This means that $y_k = x_{n(k)}$ where $n(k)$ is an increasing function of k so $n(k) \longrightarrow \infty$ as $k \longrightarrow \infty$. Suppose $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Prove $x_n \longrightarrow y$ as $n \longrightarrow \infty$.

Suppose $\epsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence there is an integer N so that $\|x_n - x_m\| < \epsilon/2$ in $n,m \geqq N$. Since $y_n \longrightarrow y$ as $n \longrightarrow \infty$ there in an integer N_1 so that $\|x_{n(k)} - y\| < \epsilon/2$ for $k \geqq N_1$. Since $n(k) \longrightarrow \infty$ as $k \longrightarrow \infty$ there is an integer N_2 so that so that $n(k) \geqq N$ for $k \geqq N_2$. Let $m = \max(N_1,N_2)$. Suppose $n \geqq N$. Then we have

$$\|x_n - y\| \leqq \|x_n - x_{n(m)} + x_{n(m)} - y\|$$

$$\leqq \|x_n - x_{n(m)}\| + \|x_{n(m)} - y\| < \epsilon/2 + \epsilon/2 = \epsilon$$

so $\|x_n - y\| \longrightarrow 0$ as $n \longrightarrow \infty$.

2. Suppose I is the set of positive integers and x_n is a sequence of real numbers. When we write

$$y = \sum_{k \in I} x_k$$

we mean that for every $\epsilon > 0$ there is a finite set S of positive integers so that if T is a finite set of positive integers containing S then

$$\left| y - \sum_{k \in T} x_k \right| < \epsilon$$

Show that y is the sum of the x_k in this sense if and only if the sum of the x_k is absolutely convergent and converges to y.

Suppose y is the sum of the x_k as stated above. We show that

$$K = \sup\{\sum_{k\in T} |x_k| : T = \text{finite set of integers}\} < \infty$$

Suppose K = ∞ (so there is no upper bound of the sums over finite sets). For any finite set S_1 we can decompose it into two sets T and S where $x_k \geq 0$ for $k \in T$ and $x_k < 0$ for $k \in S$. The we see

$$K = \sup\{ \sum_{k\in T} x_k - \sum_{k\in S} x_k : T,S \text{ finite sets of integers}\}$$

we have either

$$K_+ = \sup\{\sum_{k\in T} x_k : T = \text{finite set of integers}\} = \infty$$

or

$$K_- = \sup\{\sum_{k\in T} - x_k : T = \text{finite set of integers}\} = \infty$$

Suppose $K_- = \infty$ and $\epsilon > 0$. Since the sum of the x_k is y in the sense described above we have there is a finite set S so that if T is any finite set containing T then

$$\left| y - \sum_{k\in T} x_k \right| < \epsilon$$

Since $K_- = \infty$ it is clear that the sup of over the complement of S is also infinite so

$$K_-(S^c) = \sup\{\sum_{k\in T} - x_k : T = \text{finite set of integers, } T \subset S^c\} = \infty.$$

But now we see this contridicts the convergence of the sum to y since there is a finite set $T_1 \subset S^c$ so that

$$\sum_{k\in T_1} - x_k > 2\epsilon + 1$$

which implies

$$\left|y - \sum_{k \in S \cup T_1} x_k\right| = \left|y - \sum_{k \in S} x_k - \sum_{k \in T_1} x_k\right| = \left|\sum_{k \in T_1} x_k - y + \sum_{k \in T_1} x_k\right|$$

$$\geq \left|\sum_{k \in T_1} x_k\right| - \left|y - \sum_{k \in S \cup T_1} x_k\right| \geq 2\epsilon + 2 - \epsilon = \epsilon + 2$$

and this contradicts the converges of the sum to y. A similar argument shows K_+ is finite so K is finite. Now we show the sum of the $|x_k|$ converges to K in the sense above. Suppose $\epsilon > 0$. Since K is the sup over all finite sets there is a finite set S so that

$$\sum_{k \in S} |x_k| > K - \epsilon$$

Now suppose T is any finite set of integers containing S. Then

$$\left|K - \sum_{k \in T} |x_k|\right| \leq K - \sum_{k \in T} |x_k| \leq K - \sum_{k \in S} |x_k| < K - (K - \epsilon) = \epsilon$$

So the sum of the absolute values converges to K.

Now suppose the sum of the $|x_k|$ converges to K. Then from what we have just shown if follows that

$$K = \sup\{\sum_{k \in T} |x_k| : T = \text{finite set of integers}\} < \infty.$$

Let $x_k^+ = \max(x_k, 0)$ and $x_k^- = \max(-x_k, 0)$. Since $|x_k| = \max(x_k^+, x_k^-)$ if follows that

$$K_+ = \sup\{\sum_{k \in T} x_k^+ : T = \text{finite set of integers}\} \leq K$$

$$K_- = \sup\{\sum_{k \in T} x_k^- : T = \text{finite set of integers}\} \leq K.$$

And by what we have shown above we have K_+ is the sum of the x_k^+ and K_- is the sum of the x_k^- in the sense stated above. Now we show that $K_+ - K_-$ is the sum of the x_k. Suppose $\epsilon > 0$. Then there are sets

S_+ and S_- is that if $T_+ \supset S_+$ and $T_- \supset S_-$ are finite sets of positive integers then the sum of the x_k^+ over the T_+ is within $\epsilon/2$ of K_+ and the sum of the x_k^- over T_- is within $\epsilon/2$ of K_-. Let $S = S_+ \cup S_-$ and suppose $T \supset S$ is a finite set of positive integers. Note we have $x_k = x_k^+ - x_k^-$ for each positive integer k. So we have

$$\begin{aligned} \left|K_+ - K_- - \sum_{k\in T} x_k\right| &= \left|K_+ - \sum_{k\in T} x_k^+ - K_- + \sum_{k\in T} x_k^-\right| \\ &\leq \left|K_+ - \sum_{k\in T} x_k^+\right| + \left|K_- - \sum_{k\in T} x_k^-\right| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

3. A point x is an extreme point of a convex set S if x is not of the form $x = tx_1 + (1 - t)x_2$ with $x_1, x_2 \in S$ and $x_1 \neq x_2$ and $0 < t < 1$ (i.e. x is not in the interior of a line segment if S). Show that a unit vector f in a Hilbert space is an extreme point of the unit ball.

Suppose f and g are vectors in the unit ball of a Hilbert space $\mathfrak{H}$ and $h = tf + (1 - t)g$ with $t \in (0,1)$. Suppose h is a unit vector. We have

$$\begin{aligned} 1 = (h,h) &= t^2\|f\|^2 + 2t(1 - t)\mathrm{Re}(f,g) + (1 - t)^2\|g\|^2 \\ &\leq 1 - 2t + 2t^2 + 2t(1 - t)\mathrm{Re}(f,g). \end{aligned}$$

Hence, we have $\mathrm{Re}(f,g) \geq 1$ and since by the Schwarz inequality we have $|(f,g)| \leq \|f\|\|g\| \leq 1$ we have $\mathrm{Re}(f,g) = 1$ and since

$$1 = \mathrm{Re}(f,g) \leq \|f\|\|g\| \leq \|f\|$$

we have $1 \leq \|f\| \leq 1$ so $\|f\| = 1$. Then we have from the above inequality that $1 \leq \|g\|$ and since g is in the unit ball of $\mathfrak{H}$

we have $\|g\| = 1$. Then we have

$$\|f - g\|^2 = ((f - g),(f - g)) = \|f\|^2 - 2\mathrm{Re}(f,g) + \|g\|^2 = 2 - 2 = 0$$

so $f = g$ and h is an extremal element of the unit ball.

4. A Banach X space is said to be uniformly convex if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ so that if x and y are in the unit ball of X (i.e. $\|x\| \leq 1$ and $\|y\| \leq 1$). and $\|\frac{1}{2}(x + y)\| > 1 - \delta$. Then $\|x - y\| < \epsilon$. Prove that a Hilbert space is uniformly convex. (i.e. given $\epsilon > 0$ find $\delta > 0$).

Suppose f and g are vectors in the unit ball of a Hilbert space. By the parallelogram law we have

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

so

$$\|f - g\|^2 \leq 4 - 4\|\tfrac{1}{2}(f + g)\|^2.$$

Suppose $\epsilon > 0$. Let

$$\delta = 1 - \sqrt{1 - \epsilon^2/4}.$$

Then if $\|\frac{1}{2}(f + g)\| > 1 - \delta$ we have

$$\|f - g\|^2 \leq 4 - 4\|\tfrac{1}{2}(f + g)\|^2 < 4 - 4(1 - \delta)^2 = 8\delta - 4\delta^2 = 4\delta(2 - \delta)$$

$$= 4(1 - \sqrt{1 - \epsilon^2/4})(1 + \sqrt{1 - \epsilon^2/4}) = 4(1 - (1 - \epsilon^2/4)) = \epsilon^2$$

so $\|f - g\| < \epsilon$.

5. Consider the real linear space of all continuous real valued functions on $[-1,1]$ with inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\, dx.$$

Consider the functions, $f_n(x) = x^n$ for $n = 0,1,2,3,4$. Use the Schmidt orthogonalization procedure to produce an orthonomal sequence $\{g_0, g_1, \cdots, g_4\}$.

$$g_0(x) = \frac{1}{\sqrt{2}}\, x, \qquad g_1(x) = \sqrt{3/2}\; x,$$

$$g_2(x) = \frac{3}{2}\sqrt{5/2}\;(x^2 - 1/3),$$

$$g_3(x) = \frac{5}{2}\sqrt{7/2}\;(x^3 - \frac{3}{5}x),$$

$$g_4(x) = \frac{3\cdot 5\cdot 7}{2\sqrt{59}}\;(x^4 - \frac{6}{7}x^2 + \frac{3}{35})$$

6. Suppose $\{f_1, \cdots, f_n\}$ is a finite set of vectors. Show the vectors $\{f_1, \cdots, f_n\}$ are linearly independent if and only if the $(n \times n)$-matrix $M_{ij} = (f_i, f_j)$ is invertable. You may use the fact that M is not invertable if and only if there is a non zero vector $(c_1, \cdots, c_n) \in \mathbb{C}^n$ so that

$$\sum_{k=1}^{n} M_{ik}\, c_k = 0$$

for $i = 1, \cdots, n$.

If M is not invertable then there is a n-tuple $(c_1, \cdots, c_n)$ so that not all the c_i are zero and the above sums are zero. Let $f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$. Then we have

$$(f,f) = \sum_{i,k=1}^{n} \bar{c}_i c_k\, (f_i, f_k) = \sum_{i=1}^{n} \bar{c}_i \sum_{k=1}^{n} M_{ik}\, c_k = 0$$

so $f = 0$ and the f_i are linearly dependent.

Conversely, suppose the f_i are linearly dependent. Then there is an n-tuple $(c_1, \cdots, c_n)$ so that not all the c_i are zero so that

$f = c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$. Then for each $i = 1,\cdots,n$ we have

$$(f_i,f) = \sum_{k=1}^{n} c_k(f_i,f_k) = \sum_{k=1}^{n} M_{ik}\, c_k = 0.$$

Then from the above we have M is not invertable.

8. Let $\mathfrak{H}_0$ be the real linear space of continuously differentiable functions on [0,1] (so f has a derivative f′ which is continuous) which satisfy the boundary condition $f(0) = f(1) = 0$. We define an inner product on $\mathfrak{H}_0$ given by

$$(f,g) = \int_0^1 f'(x)g'(x)\,dx$$

Let $\mathfrak{H}$ be the completion of $\mathfrak{H}_0$. Show that each f in the completion is represented by a continuous function f(x) on [0,1] with $f(0) = f(1) = 0$ by proving that if f_n is a Cauchy sequence if $\mathfrak{H}_0$ then there is a unique continuous function f so that $f_n(x) \to f(x)$ as $n \to \infty$ and the convergence is uniform. (Hint. Show that there is a constant K so that if $x \in [0,1]$ and $f \in \mathfrak{H}_0$ then $|f(x)| \leq K\|f\|$). Let S be the set of functions on [0,1] which are limits of Cauchy sequences $f_n \in \mathfrak{H}_0$ in the unit ball so $\|f_n\| \leq 1$ for all n. Show the functions $f \in S$ are uniformly equicontinuous meaning that for each $\epsilon > 0$ there is a $\delta > 0$ so that if $f \in S$ and $x,y \in [0,1]$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. The uniform or L^∞ norm on C[0,1] is given by $\|f\|_\infty = \sup\{|f(x)|: x \in [0,1]\}$. By Ascoli's Theorem we have S is conditionally compact (i.e. the closure of in the uniform topology is compact in the uniform topology). Extra credit. Is S closed in the uniform topology? Let S_∞ be the set of functions in which are limits of Cauchy sequences $f_n \in \mathfrak{H}$ (i.e. we do not require

$\|f_n\| \leq 1$). Is S_∞ closed in the uniform topology?

Suppose $f \in \mathfrak{H}_0$ and $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$. Then we have

$$f(x_k) - f(x_{k-1}) = \int_{x_{k-1}}^{x_k} f'(x)\, dx$$

Since

$$\langle h,g\rangle = \int_{x_{k-1}}^{x_k} h(x)g(x)\, dx$$

is a positive bilinear form on continuous functions on [0,1] we have by the Schwarz inequality we have $|\langle h,g\rangle|^2 \leq \langle h.h\rangle\langle g,g\rangle$. Let $h_k(x) = 1$ for $x \in [x_{k-1},x_X]$ and $h_k(x) = 0$ otherwise. Then we have

$$|f(x_k) - f(x_{k-1})|^2 = |\langle f',h_k\rangle|^2 \leq \langle f',f'\rangle\langle h_k,h_k\rangle$$
$$= (x_k - x_{k-1}) \int_{x_{k-1}}^{x_k} f'(x)f'(x)\, dx$$

Hence, we have

$$\sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|^2}{x_k - x_{k-1}} \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f'(x)f'(x)\, dx$$
$$= \int_0^1 f'(x)f'(x)\, dx = \|f\|^2$$

Suppose $x \in (0,1)$. Nore for $f \in \mathfrak{H}_0$ we have $f(0) = f(1) = 0$ so from the above we have

$$\frac{|f(x)|^2}{x} + \frac{|f(x)|^2}{1 - x} \leq \|f\|^2$$

so

$$|f(x)| \leq \sqrt{x(1 - x)}\, \|f\| \leq \tfrac{1}{2}\|f\|$$

Suppose $\{f_n\}$ is a Cauchy sequence in $\mathfrak{H}_0$ we have

$$|f_n(x) - f_m(x)| \leq \tfrac{1}{2}\|f_n - f_m\|$$

so $f_n(x)$ converges to a limit $f(x)$ for all $x \in [0,1]$ and since the convergence is uniform the limiting function f is continuous. Hence, for each $f \in \mathfrak{H}$ there is represented by a unique continuous function f on [0,1]. Let S be the unit ball of $\mathfrak{H}$. Let Q be the set of continuous real valued functions f on [0,1] so that $f(0) = f(1) = 0$ and for every partition

$$\pi = (0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1)$$

of [0,1] we have

$$\sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|^2}{x_k - x_{k-1}} = \|f\|_\pi^2 \leq 1.$$

We will show that S = Q. We have show above that $S \subset Q$.

We have for $0 \leq x < y < z \leq 1$ that

$$\frac{|f(y) - f(x)|^2}{y - x} + \frac{|f(z) - f(y)|^2}{z - y} \leq \frac{|f(z) - f(x)|^2}{z - x}$$

and the equality sign holds if and only if

$$f(y) = \frac{z - y}{z - x} f(x) + \frac{y - x}{z - x} f(z)$$

Hence, if π' is a partition obtained from π by adding one point we have $\|f\|_\pi \leq \|f\|_{\pi'}$. If π_1 and π_2 are partitions of [0,1] then we say π_2 is a refinement of π_1 (denoted $\pi_1 \leq \pi_2$) if

$$\pi_1 \approx (x_1,\ldots,x_n) \subset (y_1,\ldots,y_m) \approx \pi_2.$$

Note if π_1 and π_2 are partitions of [0,1] then $\pi = \pi_1 \vee \pi_2$ whose points are the union of the points of π_1 and π_2 is least upper bound of π_1 and π_2 in that if $\pi' \geqq \pi_1$ and $\pi' \geqq \pi_2$ then $\pi' \geqq \pi_1 \vee \pi_2$. Note if $\pi' \geqq \pi$ then $\|f\|_{\pi'} \geqq \|f\|_{\pi}$. Now if $f \in \mathfrak{H}$ and π is a partition of [0,1] then we denote by f_π the function given by

$$f_\pi(x) = \frac{x_k - x}{x_k - x_{k-1}} f(x_{k-1}) + \frac{x - x_{k-1}}{x_k - x_{k-1}} f(x_k)$$

for $x \in (x_{k-1}, x_k]$. Note

$$f'_\pi(x) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

for $x \in (x_{k-1}, x_k)$. It follow then that $\|f_\pi\| = \|f\|_\pi$. Now if π and π' are partitions of [0,1] and $\pi' \geqq \pi$ one finds

$$\|f_{\pi'} - f_\pi\|^2 = \|f\|^2_{\pi'} + \|f\|^2_\pi - 2\|f\|^2_\pi = \|f\|^2_{\pi'} - \|f\|^2_\pi. \qquad (\&)$$

To see this consider the case where π' is obtained from π by adding a single point. Then one finds the above formula. Since any refinement of π can be obtained by adding a finite number of points to π we obtain the above formula. Now suppose $f \in Q$ and let

$$K = \sup\{\|f\|_\pi : \pi \text{ a partition of } [0,1]\}.$$

Note since $f \in Q$ we have $K \leqq 1$. Let π_k be partitions of [0,1] so that $\|f\|_{\pi_k} > K - 1/k^2$ and $|\pi_k| < 1/k^2$ where ($|\pi_k|$ = width of the largest interval of π_k) for $k = 1, 2, \cdots\cdot$. Suppose π and π' are partitions of [0,1] and $\pi'' = \pi \vee \pi'$ is their common refinement. Then we have

$$\|f_{\pi'} - f_\pi\| = \|f_{\pi'} - f_{\pi''} + f_{\pi''} - f_\pi\| \leqq \|f_{\pi'} - f_{\pi''}\| + \|f_{\pi''} - f_\pi\|$$

and from (&) above we then have

$$\|f_{\pi'} - f_\pi\| \leqq \sqrt{\|f\|^2_{\pi''} - \|f\|^2_{\pi'}} + \sqrt{\|f\|^2_{\pi''} - \|f\|^2_\pi}$$

$$\|f_{\pi'} - f_\pi\| \le \sqrt{(\|f\|_{\pi''} + \|f\|_{\pi'})(\|f\|_{\pi''} - \|f\|_{\pi'})}$$

$$+ \sqrt{(\|f\|_{\pi''} + \|f\|_{\pi})(\|f\|_{\pi''} - \|f\|_{\pi})}$$

$$\le \sqrt{2\|f\|_{\pi''}} \left(\sqrt{\|f\|_{\pi''} - \|f\|_{\pi'}} + \sqrt{\|f\|_{\pi''} - \|f\|_{\pi}}\right)$$

$$\le \sqrt{2K} \left(\sqrt{K - \|f\|_{\pi'}} + \sqrt{K - \|f\|_{\pi}}\right)$$

Now if n and m are integers and $\pi = \pi_n$ and $\pi' = \pi_m$ then

$$\|f_{\pi'} - f_\pi\| \le \sqrt{2K}\,(1/m + 1/n)$$

so f_{π_n} for $n = 1,2,\cdots$ is a Cauchy sequence in $\mathfrak{H}$. Since $|\pi_n| \to 0$ as $n \to \infty$ one sees that $f_{\pi_n}(x)$ converges to $f(x)$ as $n \to \infty$ so f corresponds to an element of S so Q = S.

Now one sees that Q is closed in the topology of uniform convergence. If f_n is a sequence of elements of S and $f_n \to f$ uniformly then for any partition π we see that $\|f_n\|_\pi^2 \to \|f\|_\pi^2$ as $n \to \infty$ so the limiting function f satisfies $\|f\|_\pi \le 1$ for all partitions π so $f \in Q = S$ so Q is closed in the uniform topology.

Note S_∞ is not closed in the uniform topology. To see this suppose $f_n(x) = \min(\sqrt{1/2} - \sqrt{|x - \tfrac12|}, \sqrt{1/2} - \sqrt{1/n})$ for $x \in [0,1]$ and $n = 1,2,\cdots$. Now, f_n converges uniformly to $f(x) = \sqrt{1/2} - \sqrt{|x - \tfrac12|}$ for $x \in [0,1]$. One calculates that

$$\|f_n\|^2 = \int_0^1 |f_n'(x)|\,dx = 2\int_0^{1/2 - 1/n} (1/4)\,|x - \tfrac12|^{-1}\,dx = \tfrac12 \ln(n/2) \to \infty$$

as $n \to \infty$. Let R be the set of partitions π so that the interval $[\tfrac12 - 1/n, \tfrac12 + 1/n]$ is one of the intervals (so this interval is never subdivided) then

$$\sup(\|f\|_\pi^2 : \pi \in R\} = \tfrac12 \ln(n/2)$$

so it follows that $\sup\{\|f\|_{\pi}^{2}: \pi$ any partition of $[0,1]\} = \infty$ so f does not correspond to an element of $\mathfrak{H}$. Hence, S_{∞} is not closed in the uniform topology.