

Answers for Homework 3 Due Thurs Oct. 26.

1. Suppose E is a projection on a Banach space X so $E \in \mathfrak{B}(X)$ and $E^2 = E$. Suppose $E \neq 0$ and $E \neq I$. Show the spectrum of E consists of the two points 0 and 1. Suppose \mathfrak{H} is a Hilbert space and $E \in \mathfrak{B}(\mathfrak{H})$ is an hermitian projection (so $E = E^* = E^2$). Show $\|E\| = 1$.

Suppose $\lambda \in \mathbb{C}$. Then $(E - \lambda I)^{-1} = (\lambda(1 - \lambda))^{-1}(E + (\lambda - 1)I)$ so if $\lambda \in \sigma(E)$ we have $\lambda = 0$ or 1. Since $E(I - E)x = 0$ for all $x \in X$ and $(I - E) \neq 0$ it follows E has no inverse so $0 \in \sigma(E)$ and since $(I - E)Ex = 0$ for all $x \in X$ and $E \neq 0$ it follows that $E - I$ has no inverse and $1 \in \sigma(E)$.

2. Let A be the (2×2) -matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

which defines a linear operator on the Hilbert space \mathbb{C}^2 . Compute the spectrum of A , $\sigma(A)$ and the norm $\|A\|$.

Since $(A - \lambda I)$ has an inverse if and only if the determinant of $(A - \lambda I)$ $\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)$ is non zero it follows the spectrum of A is the set $\{1, 3\}$. We have the $\|A\|^2 = \|A^*A\|$. Since A^*A is hermitian we have $\|A^*A\| =$ the maximum eigenvalue of A^*A and the eigenvalues of A^*A are the roots of $\det(A^*A - \lambda I) = 0$. Now

$$A^*A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 1$$

so the roots are $3 \pm \sqrt{8}$ so $\|A^*A\| = 3 + \sqrt{8}$. So $\|A\| = (3 + \sqrt{8})^{1/2}$.

3. Suppose $\mathfrak{H} = L^2(2, 3)$ so \mathfrak{H} consists of the square integrable functions on $[2, 3]$. Let $Af(x) = xf(x)$. Compute $\|A\|$ and the spectrum of A .

If $\lambda \notin [2,3]$ then $((A - \lambda I)^{-1}f)(x) = (x - \lambda)^{-1}f(x)$ for $f \in \mathfrak{H}$ and $x \in [2,3]$. If $\lambda \in (2,3)$ let $h_n(x) = \max(0, 1 - n|x - \lambda|)$ for $n = 1, 2, \dots$. Let

$$c_n = (h_n, h_n) = \int_2^3 h_n(x)^2 dx = \int_{\lambda-1/n}^{\lambda+1/n} (1 - n|x - \lambda|)^2 dx = 2/3n$$

for n sufficiently large. Let $f_n = c_n^{-1/2}h_n$ so $\|f_n\| = 1$ for $n = 1, 2, \dots$. Then

$$\begin{aligned} \|(A - \lambda I)f_n\|^2 &= (1/c_n) \int_2^3 (x - \lambda)^2 h_n(x)^2 dx \\ &= (1/c_n) \int_{\lambda-1/n}^{\lambda+1/n} (x - \lambda)^2 (1 - n|x - \lambda|)^2 dx \\ &= (1/c_n)(6n^2)^{-1} = \frac{1}{4n} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for n sufficiently large. So $(A - \lambda I)$ does not have an inverse for $\lambda \in (2,3)$. Since the spectrum is closed we have $\sigma(A) = [2,3]$.

4. Consider the set X of continuous real valued functions on $[0,1]$. Consider the three norms on X given by

$$\|f\|_1 = \int_0^1 |f(x)| dx, \quad \|f\|_2^2 = \int_0^1 |f(x)|^2 dx,$$

and $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$.

Suppose $f_n \in X$ and $|f_n(x)| \leq 1$ for $x \in [0,1]$ for $n = 1, 2, \dots$. For each of the following statements either prove it is true or give a counter example. You may sketch the graph of the f_n provided the graph defines the functions in question.

- If $\|f_n\|_1 \rightarrow 0$ then $\|f_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$.
- If $\|f_n\|_1 \rightarrow 0$ then $\|f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

- c) If $\|f_n\|_2 \rightarrow 0$ then $\|f_n\|_1 \rightarrow 0$ as $n \rightarrow \infty$.
 d) If $\|f_n\|_2 \rightarrow 0$ then $\|f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.
 e) If $\|f_n\|_\infty \rightarrow 0$ then $\|f_n\|_1 \rightarrow 0$ as $n \rightarrow \infty$.
 f) If $\|f_n\|_\infty \rightarrow 0$ then $\|f_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$.

a) Since $|f_n(x)| \leq 1$ we have $|f_n(x)|^2 \leq |f_n(x)|$ so

$$\|f_n\|_2^2 = \int_0^1 |f_n(x)|^2 dx \leq \int_0^1 |f_n(x)| dx = \|f_n\|_1$$

so (a) is true.

b) Let $f_n(x) = \max(0, 1 - nx)$ for $x \in [0, 1]$. We have $\|f_n\|_1 = 2/n$ and $\|f_n\|_\infty = 1$. So (b) is false.

c) Let $g_n(x) = f_n(x)/|f_n(x)|$ when $f_n(x) \neq 0$ and let $g_n(x) = 0$ when $f_n(x) = 0$ for $n = 1, 2, \dots$. Then from the Schwarz' inequality we have

$$\|f_n\|_1 = \int_0^1 f_n(x)g_n(x) dx \leq \|f_n\|_2 \|g_n\| \leq \|f_n\|_2$$

so (c) is true.

d). Same example as (b) shows (d) is false.

e). We have

$$\|f_n\|_1 = \int_0^1 |f_n(x)| dx \leq \int_0^1 \|f_n\|_\infty dx \leq \|f_n\|_\infty$$

so (e) is true.

f). We have

$$\|f_n\|_2^2 = \int_0^1 |f_n(x)|^2 dx \leq \int_0^1 \|f_n\|_\infty^2 dx = \|f_n\|_\infty^2$$

so (f) is true.

5. Do 4 with $[0, 1]$ replaced by $[0, \infty)$.

a). We have (a) is true by the same argument as 4a.

b). We have (b) is false by the same counter example as 4b.

c). Let $f_n(x) = \max(0, 2/n(1 - x/n))$ for $n = 1, 2, \dots$. Then we have

$$\|f_n\|_2 = \sqrt{4/3} n^{-1/2} \quad \text{and} \quad \|f_n\|_1 = 1$$

so (c) is false.

d) We have (d) is false by the same counter examples as 4d.

e) We have (e) is false by the same counter example as 5c.

f) Let $f_n(x) = \sqrt{\max(0, 2/n(1 - x/n))}$ for $n = 1, 2, \dots$. Then we have $\|f_n\|_\infty = \sqrt{2/n}$ and $\|f_n\|_2 = 1$ so (f) is false.

6. Give an example of a sequence of continuous real valued functions on $[0, 1]$ so that $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in [0, 1]$ but $\|f_n\|_2$ does not converge to zero.

Let $f_n(x) = \sqrt{n^2 \max(0, 1 - n^2(x - 1/n)^2)}$ for $n = 1, 2, \dots$. Note $f_n(x) = 0$ for $x \geq 2/n$ and $f(0) = 0$ so $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Note $\|f_n\|_2 = 1$ for $n = 1, 2, \dots$ so $\|f_n\|_2$ does not converge to zero.

7. Let X be as in problem 4 and let A be the mapping of X into itself given by

$$(Af)(x) = h(x) \int_0^1 f(t) dt$$

where h is a continuous function on $[0, 1]$. Let $\|A\|_1, \|A\|_2$ and $\|A\|_\infty$ be the norm of A as an operator from X into X with the three norms defined in problem 6. Compute these three norms.

$$\|A\|_1 = \|h\|_1 \quad \text{max occurs for } f(x) = 1 \text{ on } [0, 1]$$

$$\|A\|_2 = \|h\|_2 \quad \text{max occurs for } f(x) = 1 \text{ on } [0, 1]$$

$$\|A\|_\infty = \|h\|_\infty \quad \text{max occurs for } f(x) = 1 \text{ on } [0, 1]$$

8. Assume the set up of problem 4 and let A be the operator on X given by

$$(Af)(x) = \int_0^x f(t) dt.$$

Compute $\|A\|_1$ and $\|A\|_\infty$.

We compute $\|A\|_1$. For $f \in X$ we have

$$|(Af)(x)| \leq \int_0^x |f(t)| dt \leq \int_0^1 |f(t)| dt = \|f\|_1$$

so $\|Af\|_1 \leq \|f\|_1$. Let $f_n(x) = 2\max(0, 1 - x/n)$ for $n = 1, 2, \dots$.

We have $\|f_n\|_1 = 1$ and $Af_n(x) = 1$ for $x \in [1/n, 1]$. Hence,

$$\|Af_n\|_1 \geq \int_{1/n}^1 dx = (1 - 1/n) \rightarrow 1$$

as $n \rightarrow \infty$ so $\|A\|_1 = 1$.

We compute $\|A\|_\infty$. We have

$$|Af(x)| \leq \int_0^x \|f\|_\infty dx = x\|f\|_\infty$$

so $\|Af\|_\infty \leq \|f\|_\infty$ and for $f(x) = 1$ we have $(Af)(x) = x$ so $\|A\|_\infty = 1$.

9. Let \mathfrak{S}_1 be the space of continuous real valued functions of on $[0,1]$ with inner product

$$(f, g) = \int_0^1 f(x)g(x) dx$$

and let \mathfrak{S} be the real Hilbert space obtained by taking the completion of \mathfrak{S}_1 . Let A be the operator on \mathfrak{S}_1 give by

$$(Af)(x) = \int_0^x f(t) dt$$

Show A has a unique norm continuous extension to a bounded operator on \mathfrak{S} . We also denote this extension by A .

First note that from the Schwarz's inequality we have

$$|(Af)(x)|^2 \leq \int_0^x |f(t)|^2 dt \int_0^x dt \leq x \|f\|^2$$

So

$$\|Af\|^2 = \int_0^1 |(Af)(x)|^2 dx \leq \|f\|^2 \int_0^1 x dx = \frac{1}{2} \|f\|^2$$

Suppose $f \in \mathfrak{H}$. Since the continuous functions are dense in \mathfrak{H} there is a sequence of continuous functions f_n so that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$. Then since $\|Af_n - Af_m\| \leq 2^{-1/2} \|f_n - f_m\| \rightarrow 0$ as $n, m \rightarrow \infty$ and \mathfrak{H} is complete there is a unique vector $g \in \mathfrak{H}$ so that $Af_n \rightarrow g$ in norm as $n \rightarrow \infty$. Suppose h_n is a second sequence converging to f as $n \rightarrow \infty$ we and $Ah_n \rightarrow g_1$ as $n \rightarrow \infty$. Then we have

$$\|g - g_1\| = \lim_{n \rightarrow \infty} \|Af_n - Ah_n\| \leq \lim_{n \rightarrow \infty} 2^{-1/2} \|f_n - h_n\| = 0$$

so $g_1 = g$ and the definition of Af depends only on f and not on which approximating sequence you choose. If $f, g \in \mathfrak{H}$ and f_n and g_n are sequences of continuous functions which converge in \mathfrak{H} to f and g and if $a \in \mathbb{C}$ then $af_n + g_n$ converges in norm to $af + g$ and, hence, we have

$$A(af + g) = \lim_{n \rightarrow \infty} A(af_n + g_n) = \lim_{n \rightarrow \infty} (aAf_n + Ag_n) = aAf + Ag$$

so A is a linear operator. Let if $K = \sup\{\|Af\| : f \in \mathfrak{H}_1 \text{ and } \|f\| \leq 1\}$. We show $\|A\| = K$. Clearly we have $\|A\| \leq K$. Suppose $f \in \mathfrak{H}$ and $f_n \in \mathfrak{H}_1$ is a sequence converging to f in norm. Then

$$\|Af\| = \lim_{n \rightarrow \infty} \|Af_n\| \leq K \lim_{n \rightarrow \infty} \|f_n\| = K\|f\|,$$

Hence, $\|A\| \leq K$ so $\|A\| = K \leq 2^{-1/2}$.

10. Prove the operator A of problem 9 is compact. Hint. Suppose $f_n \rightarrow 0$ in \mathfrak{H} weakly. Show $Af_n(x) \rightarrow 0$ for each $x \in [0,1]$. Show the functions Af_n are equicontinuous (note that $\|f_n\|$ is bounded by the principle of uniform boundedness).

Suppose $f_n \in \mathfrak{H}$ and $f_n \rightarrow 0$ weakly as $n \rightarrow \infty$. By the principle of uniform boundedness there is a constant M so that $\|f_n\| \leq M$ for all $n = 1, 2, \dots$. Consider the functions $g_n(x) = (Af_n)(x)$. Suppose $0 \leq x < y \leq 1$. Then we have from the Schwarz inequality that

$$|g_n(x) - g_n(y)|^2 = \left| \int_x^y f_n(t) dt \right|^2 \leq \int_x^y |f_n(t)|^2 dt \int_x^y dt \leq |x - y| \|f_n\|^2$$

for all $n = 1, 2, \dots$. Hence, we have for all $x, y \in [0,1]$ that

$$|g_n(x) - g_n(y)| \leq \sqrt{|x - y|} M$$

for all $n = 1, 2, \dots$. Since $g_n(0) = 0$ we have

$$|g_n(x)| \leq M$$

for all $x \in [0,1]$ and $n = 1, 2, \dots$. Hence, the functions g_n are uniformly bounded and uniformly equicontinuous and converge pointwise to zero for all $x \in [0,1]$ so we have $\|g_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ where $\|h\|_\infty = \sup\{|h(x)| : x \in [0,1]\}$. Then we have

$$\|g_n\|^2 = \int_0^1 |g(x)|^2 dx \leq \int_0^1 \|g_n\|_\infty^2 dx = \|g_n\|_\infty^2 \rightarrow 0$$

12. Compute A^*f for $f \in \mathfrak{H}_1$. (Integration by parts.)

Suppose $f, g \in \mathfrak{H}_1$. Then we have using integration by parts

$$(f, A^*g) = (Af, g) = \int_0^1 \int_0^x \overline{f(t)} g(x) dt dx$$

$$\begin{aligned}
&= - \int_0^1 \int_0^x \overline{f(t)} dt \frac{d}{dx} \int_x^1 g(s) ds dx \\
&= - \int_0^x \overline{f(t)} dt \int_x^1 g(s) ds \Big|_{x=0}^{x=1} + \int_0^1 \frac{d}{dx} \int_0^x \overline{f(t)} dt \int_x^1 g(s) ds dx \\
&= \int_0^1 \overline{f(x)} \int_x^1 f(t) dt dx
\end{aligned}$$

Let A_1 be defined on \mathfrak{H}_1 by the formula

$$(A_1 f)(x) = \int_x^1 f(t) dt$$

We have $(f, A^*g) = (f, A_1g)$ for all $f, g \in \mathfrak{H}_1$. Since \mathfrak{H}_1 is dense in \mathfrak{H} we have $A^*g = A_1g$ for all $g \in \mathfrak{H}_1$.