Homework for second test to be given Thursday Oct. 16

Definition. $\lim_{n\to\infty} x_n = a$ or $x_n \to a$ as $n \to \infty$ means that for every $\epsilon > 0$ there is an integer N so that $|x_n - a| < \epsilon$ if $n \geq N$.

Definition. A sequence $\{x_n\}$ is a Cauchy sequence if for every $\epsilon > 0$ there is an integer N so that if $n, m \geq N$ then $|x_n - x_m| < \epsilon$.

Theorem. Every Cauchy sequence converges to a unique limit (i.e., if $\{x_n\}$ is a Cauchy sequence there is a unique real number a so that $x_n \to a$ as $n \to \infty$.

Theorem. If $\{x_n\}$ in an increasing sequence (i.e. $x_1 \leq x_2 \leq x_3 \leq \cdots\cdot$) which is bounded above (i.e. there is a number b so that $b \geq x_n$ for all $n = 1,2,\cdots\cdot$ then $x_n \to a$ as $n \to \infty$ and $a \leq b$.

Theorem. If $\{x_n\}$ in a decreasing sequence (i.e. $x_1 \geq x_2 \geq x_3 \geq \cdots\cdot$) which is bounded below (i.e. there is a number b so that $b \leq x_n$ for all $n = 1,2,\cdots\cdot$ then $x_n \to a$ as $n \to \infty$ and $a \geq b$.

Definition. If S is a set of real numbers we say b is an upper bound if $x \leq b$ for all $x \in S$. We say b is the least upper bound for S if b is an upper bound and b' is also an upper bound then $b \leq b'$. (i.e. b is a lower bound for the set of upper bounds of S)

Definition. If S is a set of real numbers we say b is an lower bound if $x \geq b$ for all $x \in S$. We say b is the greatest lower bound for S if b is a lower bound and b' is also a lower bound then $b \geq b'$. (i.e. b is an upper bound for the set of lower bounds of S)

Theorem. Every non empty set with an upper bound has a least upper bound. Every non empty set with a lower bound has a greatest lower bound.

Definition. If $\{x_n\}$ is a sequence of real numbers we define $\limsup x_n = \lim_{n\to\infty} b_n$ where b_n is the least upper bound of the set $\{x_n, x_{n+1}, x_{n+2}, \cdots\cdot\}$ and if these sets have no upper bound we define $\limsup x_n = \infty$. We define $\liminf x_n = \lim_{n\to\infty} a_n$ where a_n is the greatest lower bound of the set $\{x_n, x_{n+1}, x_{n+2}, \cdots\cdot\}$ and if there is no lower bound we define $\liminf x_n = -\infty$.

Definition. Suppose f is a real valued function defined on a domain D and the open interval $a < x < b$ denoted (a,b) is contained in D (i.e. $(a,b) \subset D$). We say f(x) approaches b as x approaches $x_0 \in D$ if for every real number $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in D$, $x \neq x_0$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. This may be denoted as follows

$$\lim_{x \to x_0} f(x) = b \qquad \text{or} \qquad f(x) \to b \quad \text{as } x \to x_0$$

We say f(x) approaches b as x approaches $x_0 \in D$ from the right if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in D$ and $0 < x - x_0 < \delta$ then $|f(x) - f(x_0)| < \epsilon$. This may be denoted as follows

$$\lim_{x \to x_0+} f(x) = b \qquad \text{or} \qquad f(x) \to b \quad \text{as } x \to x_0+$$

We say f(x) approaches b as x approaches $x_0 \in D$ from the left if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in D$ and $0 < x_0 - x < \delta$ then $|f(x) - f(x_0)| < \epsilon$. This may be denoted as follows

$$\lim_{x \to x_0-} f(x) = b \qquad \text{or} \qquad f(x) \to b \quad \text{as } x \to x_0-$$

Definition. Suppose f is a function defined on its domain D. We say f is continuous on D if for every $x_0 \in D$ and $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in D$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

Definition. Suppose f is a function defined on its domain D. We say f is uniformly continuous on D if for every $\epsilon > 0$ there is a $\delta > 0$ so that if $x,y \in D$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Definition. A real valued function f defined in an interval [a,b] is differentiable at the point $x \in (a,b)$ if the following limit exists.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

where in the limit we restrict the value of h so that $x + h \in [a,b]$. The limit is denoted $f'(x) = \frac{df}{dx}(x)$ is called the derivative of f at x. If f is differentiable at each point $x \in (a,b)$ we say f is differentiable in (a,b) We say f is differentiable at the end points $x = a$ or $x = b$ it the above limit exists where we make the restriction $h > 0$ at $x = a$ and $h < 0$ at $x = b$.

Homework problems.

1. Given $\epsilon > 0$ find a $\delta > 0$ so that $|2x + 3x^2 + 4x^3 + 5x^5| < \epsilon$ if $|x| < \delta$.
2. Prove $1/x \longrightarrow 4$ as $x \longrightarrow 1/4$.
3. Compute the lim sup and the lim inf of $x_n = 2 + (-1)^n(1 + 1/n)$.
4. Suppose $x_n \longrightarrow a$ as $n \longrightarrow \infty$. Prove lim sup $x_n = a$.
5. Suppose lim sup $x_n = a$ and lim inf $x_n = a$. Prove $x_n \longrightarrow a$ as $n \longrightarrow \infty$.
6. Suppose x_n is a Cauchy sequence. Suppose $x_n \geq 0$ for infinitely many positive integers n and $x_n \leq 0$ for infinitely many positive integers n. Prove $x_n \longrightarrow 0$ as $n \longrightarrow \infty$.
7. Suppose $\{x_n\}$ is a non decreasing sequence so $x_1 \leq x_2 \leq x_3 \leq x_4 \leq \cdots\cdot$ which is bounded above by c so $x_k \leq c$ for all $k = 1,2,\cdots\cdot$. Let b be the least upper bound of the set $\{x_1, x_2, \cdots\cdot\}$. Prove $x_n \longrightarrow b$ as $n \longrightarrow \infty$.
9. Suppose f is a continuous function defined in the open interval (a,b).
10. Prove the function $f(x) = x^3$ defined on the interval $[-1,1]$ is differentiable in $(-1,1)$.
11. Prove that the function $f(x) = 1/x$ defined on the interval $[1,2]$. Prove f is differentiable in $(1,2)$.
12. Is the function $f(x) = |x|$ differentiable at $x = 0$?
13. Is the function $f(x) = |x|^{3/2}$ differentiable in $(-1,1)$? If so what is its derivative?
14. Prove the function $f(x) = \sqrt{|x|}$ is uniformly continuous on $(-\infty,\infty)$.
15. Prove the function $f(x) = x^2$ is uniformly continuous on the closed interval $[0,5]$. ($x \in [0,5]$ if and only if $0 \leq x \leq 5$).
16. Prove the function $f(x) = x^2$ is not uniformly continuous on $[0,\infty)$ Note $[0,\infty) = \{x: x \geq 0\}$.
17. Let $D = [1,\infty) = \{x: x \geq 1\}$. Let $f(x) = 1/x$ on D. Prove f is uniformly continuous on D.
18. Let $D = (0,1) = \{x: 0 < x < 1\}$. Let $f(x) = 1/x$ on D. Prove f is not uniformly continuous on D.
19. Let $D = [0,\infty)$ and $f(x) = \sqrt{x}$ on D. Prove f is uniformly continuous on D.