

**Math 360 (Powers) 1½ Hour Test. Answers September 25, 2008**

1. Suppose  $x_n \geq 0$  for  $n = 1, 2, \dots$  and  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Prove  $a \geq 0$ .

Proof. Assume the hypothesis. Assume  $a < 0$ . Since  $x_n \rightarrow a$  as  $n \rightarrow \infty$  and  $-a > 0$  there is an integer  $N$  so that  $|x_n - a| < -a$  for  $n \geq N$ . Then we have  $x_N - a \leq |x_N - a| < -a$  so  $x_N < 0$ . But this is a contradiction since  $x_N \geq 0$ . Hence,  $a \geq 0$ .

2. Suppose  $x_n = (-1)^n$  for  $n = 1, 2, \dots$ . Prove the  $\{x_n\}$  does not converge.

Proof. Suppose  $x_n = (-1)^n$  for  $n = 1, 2, \dots$  and  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Then there is an integer  $N$  so that  $|x_n - a| < 1$  for  $n \geq N$ . Let  $p$  be the first even integer greater than  $N$ . Then  $|x_p - a| = |1 - a| < 1$  so  $1 - a < 1$  and  $a > 0$ . We also have  $|x_{p+1} - a| = |-1 - a| < 1$  so  $-1 < -1 - a$  and  $-a > 0$  or  $a < 0$ . The number  $a$  can not be both positive and negative so we have reached a contradiction. Hence, the sequence  $\{x_n\}$  can not converge.

3. Suppose  $x_n > 0$  and  $x_n \rightarrow 1/4$  as  $n \rightarrow \infty$ . Prove  $1/x_n \rightarrow 4$  as  $n \rightarrow \infty$ .

Proof. Suppose  $\epsilon > 0$ . Let  $\epsilon_1 = \min(1/8, \epsilon/32)$ . Since  $x_n \rightarrow 1/4$  as  $n \rightarrow \infty$  there is an integer  $N$  so that  $|x_n - 1/4| < \epsilon_1$  for  $n \geq N$ . Suppose  $n \geq N$ . then  $1/4 - x_n \leq |x_n - 1/4| < \epsilon_1 \leq 1/8$  so  $x_n > 1/4 - 1/8 = 1/8$  and we have

$$\left| \frac{1}{x_n} - 4 \right| = 4 \left| \frac{1/4}{x_n} - 1 \right| = 4 \left| \frac{1/4 - x_n}{x_n} \right| = \frac{4|x_n - 1/4|}{|x_n|} < \frac{4|x_n - 1/4|}{1/8} < 32\epsilon_1 \leq \frac{32\epsilon}{32} = \epsilon$$

4. Suppose  $\{x_n\}$  is sequence of positive numbers and  $x_n \rightarrow 2$  as  $n \rightarrow \infty$ .

Let  $y_n$  be the average of the first of  $\{x_1, \dots, x_n\}$  so  $y_n = \frac{1}{n} \sum_{k=1}^n x_k$ .

Prove  $y_n \rightarrow 2$  as  $n \rightarrow \infty$ .

Proof. Assume the hypothesis. Suppose  $\epsilon > 0$ . Since  $x_n \rightarrow 2$  as  $n \rightarrow \infty$  there is an integer  $N_1$  so that  $|x_n - 2| < \epsilon/2$  for  $n \geq N_1$ . Let  $C = \sum_{k=1}^{N_1-1} |x_k - 2|$  and let  $N_2$  be the first integer greater than  $2C/\epsilon$ .

Let  $N = \max\{N_1, N_2\}$ . Suppose  $n \geq N$ . Then we have

$$\begin{aligned} |y_n - 2| &= \left| \frac{1}{n} \sum_{k=1}^n (x_k - 2) \right| \leq \frac{1}{n} \sum_{k=1}^{N_1-1} |x_k - 2| + \frac{1}{n} \sum_{k=N_1}^n |x_k - 2| \\ &< C/n + \frac{1}{n} \sum_{k=N_1}^n \epsilon/2 = C/n + (\epsilon/2)(n+1-N_1)/n \leq C/n + \frac{1}{2}\epsilon \leq C/N_2 + \frac{1}{2}\epsilon < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

5. Suppose  $\{x_n\}$  is a Cauchy sequence and  $x_n$  does not converge to zero as  $n \rightarrow \infty$ .

Prove there is a positive real number  $\delta > 0$  and an integer  $N$  so that  $|x_n| > \delta$   $n \geq N$  and if  $n, m \geq N$  then  $x_n$  and  $x_m$  are either both positive or both negative.

Proof. Assume the hypothesis. Since  $x_n$  does not converge to zero as  $n \rightarrow \infty$  there is a number  $\epsilon > 0$  so  $|x_n| \geq \epsilon$  for infinitely many  $n$ . Let  $\delta = \epsilon/2$ . Since  $x_n$  is a Cauchy sequence there is an integer  $N$  so that  $|x_n - x_m| < \delta$  for  $n, m \geq N$ . Let  $p$  be the first integer so that  $p \geq N$  and  $|x_p| \geq \epsilon = 2\delta$ . Either  $x_p > 2\delta$  or  $x_p < -2\delta$ . Suppose  $x_p > 2\delta$ . If  $n \geq N$  we have  $x_p - x_n < \delta$  so  $x_n > x_p - \delta \geq 2\delta - \delta = \delta$ . Suppose  $x_p < -2\delta$ . If  $n \geq N$  we have  $x_n - x_p < \delta$  so  $x_n < x_p + \delta \leq -2\delta + \delta = -\delta$ .