

**Math 360 (Powers) 1½ Hour Test. Thursday February 11, 2010**

1. Suppose  $x_n \geq 0$  for  $n = 1, 2, \dots$  and  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Prove  $a \geq 0$ .  
Assume the hypothesis and  $a < 0$ . Let  $\epsilon = -a$ . Since  $x_n \rightarrow a$  as  $n \rightarrow \infty$  there is an integer  $N$  so that  $|x_n - a| < \epsilon = -a$  for  $n \geq N$ . In particular we have  $|x_N - a| < -a$  which yields  $x_N - a < -a$  or  $x_N < 0$ . But this is a contradiction since  $x_N \geq 0$  so  $a \geq 0$ .

2. Suppose  $x_n = \frac{1}{2} + \frac{1}{2}(-1)^n$  for  $n = 1, 2, \dots$  so  $x_n = 0$  for  $n$  odd and  $x_n = 1$  for  $n$  even. Prove  $\{x_n\}$  does not converge.

Proof. Assume the hypothesis and  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Then there is a positive integer  $N$  so that  $|x_n - a| < 1/2$  for  $n \geq N$ . Let  $k$  be the first even integer greater than  $N$ . Then  $|x_k - a| = |1 - a| < 1/2$  and  $|x_{k+1} - a| = |0 - a| < 1/2$ . Hence,  $a < 1/2$  and  $1 - a < 1/2$  or  $a > 1/2$ . This is impossible so the sequence does not converge.

3. Suppose  $x_n > 0$  for  $n = 1, 2, \dots$  and  $x_n \rightarrow 1/8$  as  $n \rightarrow \infty$ . We prove  $1/x_n \rightarrow 8$  as  $n \rightarrow \infty$ . Suppose  $\epsilon > 0$ . Let  $\epsilon_1 = \min(1/16, \epsilon/128)$ . Since  $x_n \rightarrow 1/8$  as  $n \rightarrow \infty$  there is an integer  $N$  so that  $|x_n - 1/8| < \epsilon_1$  for  $n \geq N$ . Suppose  $n \geq N$ . Then  $|x_n - 1/8| < \epsilon_1 \leq 1/16$  so

$$-1/16 < x_n - 1/8 \quad \text{so} \quad x_n > 1/16$$

Then we have

$$|1/x_n - 8| = \frac{|1 - 8x_n|}{|x_n|} = \frac{8|x_n - 1/8|}{|x_n|} < 128|x_n - 1/8| < \frac{128\epsilon}{128} = \epsilon$$

4. Suppose  $\{x_n\}$  is sequence of positive numbers and  $x_n \rightarrow 5$  as  $n \rightarrow \infty$ .  
 Let  $y_n$  be the average of the first of  $\{x_1, \dots, x_n\}$  so  $y_n = \frac{1}{n} \sum_{k=1}^n x_k$ .  
 Prove  $y_n \rightarrow 5$  as  $n \rightarrow \infty$ .

Assume the hypothesis. Suppose  $\epsilon > 0$ . Since  $x_n \rightarrow 5$  as  $n \rightarrow \infty$  there is a positive integer  $N_1$  so that  $|x_n - 5| < \epsilon/2$  for  $n \geq N_1$ . Let

$$K = \left| \sum_{k=1}^{N_1-1} x_k - 5 \right|$$

and let  $N_2$  be there first integer so that  $N_2 > 2K/\epsilon$ . Let  $N = \max(N_1, N_2)$ . Suppose  $n \geq N$ . Then we have

$$|y_n - 5| = \left| \frac{1}{n} \sum_{k=1}^n x_k - 5 \right| = \frac{1}{n} \left| \sum_{k=1}^{N_1-1} x_k - 5 + \sum_{k=N_1}^n x_k - 5 \right|$$

From the triangle inequality we have

$$\begin{aligned} |y_n - 5| &\leq \frac{1}{n} \left| \sum_{k=1}^{N_1-1} x_k - 5 \right| + \frac{1}{n} \sum_{k=N_1}^n |x_k - 5| < K/n + \frac{1}{n} \sum_{k=N_1}^n \epsilon/2 \\ &= K/n + \frac{n - N_1 + 1}{2n} \epsilon \leq K/n + \epsilon/2 < K/(2K\epsilon^{-1}) + \epsilon/2 = \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

5. Suppose  $\{x_n\}$  is decreasing sequence (i.e.  $x_1 > x_2 > x_3 > \dots$ ) which is bounded below. Suppose  $b$  is the greatest lower bound of the  $x_n$  (i.e.  $b \leq x_n$  for all  $n = 1, 2, \dots$  and if  $b' \leq x_n$  for all  $n = 1, 2, \dots$  then  $b' \leq b$ .) Prove  $x_n \rightarrow b$  as  $n \rightarrow \infty$ .

Assume the hypothesis. Suppose  $\epsilon > 0$ . Since  $b$  is the greatest lower bound  $b + \epsilon$  is not a lower bound. Hence there in an integer  $N$  so that  $x_N < b + \epsilon$ . Suppose  $n \geq N$ . Then we have

$$b - \epsilon < b \leq x_n \leq x_N < b + \epsilon.$$

Hence  $|x_n - b| < \epsilon$ . Q.E.D.