

Math 360 (Powers) 1½ Hour Test. Thursday November 6, 2008

1. Suppose f is strictly increasing on the closed interval $[0,1]$ so if $0 \leq x < y \leq 1$ then $f(x) < f(y)$. Prove that the $\lim_{x \rightarrow 0^+} f(x) = b$ and $f(0) \leq b$. (Note $\lim_{x \rightarrow 0^+} f(x) = b$ means for every $\epsilon > 0$ there is a $\delta > 0$ so that $|f(x) - b| < \epsilon$ if $0 < x < \delta$.)

Proof. Suppose f is strictly increasing on the closed interval $[0,1]$. Let $S = \{f(x) : \text{for } 0 < x \leq 1\}$. Note S is not empty since $f(\frac{1}{2}) \in S$ and S is bounded below by $f(0)$. Then S has a greatest lower bound $b = \inf(S)$. We claim $f(x) \rightarrow b$ as $x \rightarrow 0^+$. Suppose $\epsilon > 0$. Since b is the greatest lower bound of S we have $b + \epsilon$ is not a lower bound so there is an $x_0 \in (0,1]$ so that $b + \epsilon > f(x_0)$. Let $\delta = x_0$. Suppose $0 < x < \delta$. Then $b \leq f(x) < f(x_0) < b + \epsilon$. Then we have $0 \leq f(x) - b < \epsilon$ so we have $|f(x) - b| < \epsilon$. Hence $f(x) \rightarrow b$ as $x \rightarrow 0^+$. ■

2. True or false. (Either show the following statements are true or give a counter example.)

- A. If $\limsup x_n = 1$ then $x_n \leq 1$ for n sufficiently large.

Counter example $x_n = 1 + 1/n$

- B. If $\limsup x_n = 1$ then $x_n > 0.99$ for n sufficiently large.

Counter example. Let $x_{2n} = 1$ and $x_{2n+1} = -1$ for $n = 1, 2, \dots$

- C. If $\limsup x_n = 1$ then $x_n < 1.01$ for n sufficiently large.

True. If $b_n = \sup(x_n, x_{n+1}, \dots)$ then there is an integer N so that $|b_n - 1| < 0.001$ for $n \geq N$. Then $x_n \leq b_N < 1.01$ for $n \geq N$.

- D. If $x_n < 1$ for all n and $\{x_n\}$ has a cluster point c then $c < 1$.

Counter example. $x_n = 1 - 1/n$ has a cluster point 1.

3. Suppose $\liminf x_n = 0$. Prove 0 is a cluster point of $\{x_n\}$.

Proof. Suppose $\liminf x_n = 0$ and zero is not a cluster point. Then there is an $\epsilon_0 > 0$ so that $|x_n| < \epsilon_0$ only finitely often. Let N_1 be one plus the largest integer so $|x_n| < \epsilon_0$. Then for $n \geq N_1$ we have either $x_n \geq \epsilon_0$ or $x_n \leq -\epsilon_0$. Suppose $x_n \leq -\epsilon_0$ infinitely often. Then $\inf(x_n, x_{n+1}, \dots) \leq -\epsilon_0$ for all n so $\liminf x_n \leq -\epsilon_0$ which violates the assumption that $\liminf x_n = 0$. Hence $x_n \leq -\epsilon_0$ only finitely often. Let N_2 be one plus the largest integer so that $x_n \leq -\epsilon_0$. Let N be the maximum of N_1 and N_2 . Then for $n \geq N$ we have $n \geq N_1$ so $|x_n| \geq \epsilon_0$ and since $n \geq N_2$ we must have $x_n \geq \epsilon_0$. Hence, $\liminf x_n \geq \epsilon_0$. This violates the assumption $\liminf x_n = 0$. We have reached a contradiction so 0 is a cluster point.

4. Prove $f(x) = x^3 - 3x + 1$ has exactly three roots.

Proof. We have $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. Then by the mean value theorem f is strictly increasing on $(-\infty, -1]$, strictly decreasing on $[-1, 1]$ and strictly increasing on $[1, \infty)$. Then f can not have more than one root in each of these intervals (else f would not be strictly increasing or strictly decreasing.) Note $f(-2) = -1$, $f(-1) = 3$, $f(1) = -1$, $f(2) = 3$. Then by the intermediate value theorem f has a root in $(-2, -1)$, a root in $(-1, 1)$ and a root in $(1, 2)$. Hence f has one and only one root in the intervals $(-\infty, -1)$, $[-1, 1]$ and $(1, \infty)$.

5. Suppose f is twice differentiable in the open interval $(-1, 3)$ and $f''(x) > 0$ for all x in the open interval $(0, 3)$. Suppose $f(0) = 0$ and $f(2) = 0$. Prove $f(1) < 0$.

Proof. Assume the hypothesis. Applying the mean value theorem to the function f and the interval $[0, 1]$ we find there is and $x_1 \in (0, 1)$ with

$$f'(x_1) = \frac{f(1) - f(0)}{1 - 0} = f(1).$$

Applying the mean value theorem to the function f and the interval $[1, 2]$ we find there is and $x_2 \in (1, 2)$ with

$$f'(x_2) = \frac{f(2) - f(1)}{2 - 1} = -f(1)$$

Since $f''(x) > 0$ for $x \in (0, 2)$ we have $f'(x)$ is strictly increasing so $f'(x_1) < f'(x_2)$. Hence $f(1) < -f(1)$ or $2f(1) < 0$ so $f(1) < 0$. ■