

Math 360 Hour Test Thrus April 8, 2010

1. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms (i.e. $a_n > 0$ for $n = 1, 2, \dots$). Suppose there is a constant K so that $K > S_n = \sum_{k=1}^n a_k$ for all $n = 1, 2, \dots$. We prove the series converges.

Let $S = \{S_1, S_2, \dots\}$ be the set of partial sums. Since K is an upper bound for S the set has a least upper bound $L = \sup(S)$. We prove the partial sums converge to L . Suppose $\epsilon > 0$. Since L is the least upper bound of S we have $L - \epsilon$ is not an upper bound for S . Hence there is as partial sum S_N so that $S_N > L - \epsilon$. Suppose $n \geq N$. Since the terms a_k of the series are positive we have

$$S_n - S_N = \sum_{k=N+1}^n a_k \geq 0.$$

Since L is an upper bound for the partial sums we have $S_n \leq L$. Hence, we have

$$L - \epsilon < S_N \leq S_n \leq L$$

so $|S_n - L| < \epsilon$ for $n \geq N$. ■

2. Suppose $x_n \rightarrow 0$ as $n \rightarrow \infty$. Let $b_n = \sup\{x_k : k \geq n\}$. We prove $b_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose $\epsilon > 0$. Since $x_n \rightarrow 0$ as $n \rightarrow \infty$ there is an integer N so that $|x_n - 0| = |x_n| < \epsilon/2$ for $n \geq N$. Suppose $n \geq N$. Then $x_k < \epsilon/2$ for $k \geq n$ (since $k \geq n \geq N$) so $\epsilon/2$ is an upper bound for the set $\{x_k : k \geq n\}$. Since b_n is the least upper bound of this set we have $b_n \leq \epsilon/2$. Since $n \geq N$ we have $|x_n| < \epsilon/2$ so $-\epsilon/2 < x_n$. Since b_n is an upper bound of the set $\{x_n : k \geq n\}$ we have $b_n \geq x_n > -\epsilon/2$. Hence, we have shown

$$-\epsilon < -\epsilon/2 < b_n \leq \epsilon/2 < \epsilon$$

which means $|b_n| < \epsilon$ for $n \geq N$. ■

3. Consider the function $f(x) = 1/x$ for $x \in [1, \infty)$. Compute the derivative of f directly from the definition of the derivative. For this problem you do not have to use a (δ, ϵ) arguments (i.e. you can use the limit theorems on the first pages without comment.)

From the definition of the derivative we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x - (x+h)}{x(x+h)} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \end{aligned}$$

Since $x \geq 1$ we have $x + h \rightarrow x \neq 0$ so

$$\lim_{h \rightarrow 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}$$

Hence, f is differentiable and $f'(x) = -1/x^2$.

4. Consider the function $f(x) = x^3 - 3x - 1$. Determine the number of roots and prove it. The derivative of f is

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

so $f'(x) > 0$ for $x \in (-\infty, -1)$, $f'(x) < 0$ for $x \in (-1, 1)$ and $f'(x) > 0$ for $x \in (1, \infty)$. Hence, f is strictly increasing in $(-\infty, -1)$, strictly decreasing in $(-1, 1)$ and strictly increasing in $(1, \infty)$. If f is strictly increasing or strictly decreasing in an interval it can have at most one root in the interval else it would not be strictly increasing or strictly decreasing. Hence, f has at most one root in each of the three intervals. Since f is continuous, $f(-2) = -3$ and $f(-1) = 1$ by the intermediate value theorem f has at least one root in $(-2, -1)$. Again since $f(-1) = 1$ and $f(1) = -3$ the intermediate value theorem assures that f has at least one root in $(-1, 1)$. Since $f(1) = -3$ and $f(2) = 1$ the intermediate value theorem assures that f has at least one root in $(1, 2)$. Since each of the three intervals have at least one root and since each of the intervals can have at most one root and since f does not have a root at $x = -1$ or $x = 1$ we have f has exactly three roots.

5. Suppose f is a continuous real valued function defined on the closed interval $[0,1]$ and f is twice differentiable and $f''(x) > 0$ in the open interval $(0,1)$ and $f(0) = 0$ and $f(1) = 1$. Prove $f(x) < x$ for x in the open interval $(0,1)$.

Assume the hypothesis. We will prove $f(x_0) < x_0$ for $x_0 \in (0,1)$. Since f is twice differentiable it is one differentiable. Applying the mean value theorem to the function f and the interval $[0,x_0]$ we find there is a $c_1 \in (0,x_0)$ so that

$$f'(c_1) = \frac{f(x_0) - f(0)}{x_0 - 0} = f(x_0)/x_0$$

Applying the mean value theorem to the function f and the interval $[x_0,1]$ we find there is a $c_2 \in (x_0,1)$ so that

$$f'(c_2) = \frac{f(1) - f(x_0)}{1 - x_0} = \frac{1 - f(x_0)}{1 - x_0}$$

Since $f''(x) > 0$ in $(0,1)$ we have by the corollary to the mean value theorem that $f'(x)$ is strictly increasing on $(0,1)$ and since $0 < c_1 < x_0 < c_2 < 1$ we have $f'(c_1) < f'(c_2)$. From the equations above we have

$$f(x_0)/x_0 < \frac{1 - f(x_0)}{1 - x_0}$$

Multiplying both sides of this inequality by $x_0(1 - x_0)$ (which is the product of two positive numbers so it is positive) we obtain

$$(1 - x_0)f(x_0) < x_0(1 - f(x_0))$$

Adding $x_0f(x_0)$ to both sides of this inequality yields

$$f(x_0) < x_0 \quad \blacksquare$$