

## Homework for third test to be given Thursday Nov. 8

**Definition.** If  $\{x_n\}$  is a sequence of real numbers then  $c$  is a cluster point if for every  $\epsilon > 0$  there are infinitely many  $n$  so that  $|x_n - c| < \epsilon$ .

**Theorem.** If  $\{x_n\}$  is a bounded sequence (i.e. there are numbers  $a$  and  $b$  so  $a \leq x_n \leq b$  for all  $n = 1, 2, \dots$ ) then  $\{x_n\}$  has a cluster point.

**Intermediate value Theorem.** Suppose  $f$  is a real valued function defined and continuous on the closed interval  $[a, b]$ . Suppose  $c$  lies between  $f(a)$  and  $f(b)$  (so  $f(a) < c < f(b)$  or  $f(b) < c < f(a)$ ). Then there is an  $x \in (a, b)$  so that  $f(x) = c$ .

**Maximum Value Theorem.** Suppose  $f$  is a continuous function defined on the closed interval  $[a, b]$ . Then  $f$  attains its maximum value at a point  $c \in [a, b]$ . This means  $f(c) \geq f(x)$  for all  $x \in [a, b]$ .

**Definition.** A real valued function  $f$  defined in an interval  $[a, b]$  is differentiable at the point  $x \in (a, b)$  if the following limit exists.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

where in the limit we restrict the value of  $h$  so that  $x+h \in [a, b]$ . The limit is denoted  $f'(x) = \frac{df}{dx}(x)$  is called the derivative of  $f$  at  $x$ . If  $f$  is differentiable at each point  $x \in (a, b)$  we say  $f$  is differentiable in  $(a, b)$ . We say  $f$  is differentiable at the end points  $x = a$  or  $x = b$  if the above limit exists where we make the restriction  $h > 0$  at  $x = a$  and  $h < 0$  at  $x = b$ .

**Mean Value Theorem.** Suppose  $f$  is defined and continuous on the closed interval  $[a, b]$  and differentiable in the open interval  $(a, b)$ . Then there is a  $c \in (a, b)$  so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Homework problems.

1. Suppose  $\limsup x_n = a$  and  $\liminf x_n = a$ . Prove  $x_n \rightarrow a$  as  $n \rightarrow \infty$ .
2. Show that if  $f$  is strictly increasing (so  $f(x) > f(y)$  for  $x > y$ ) then  $\lim_{x \rightarrow 0^+} f(x) = b$  exists and  $b \geq f(0)$ .
3. Suppose  $\limsup x_n = c$ . Prove  $c$  is a cluster point of  $\{x_n\}$ .
4. True or false. (Either show the following statements are true or give a counter example.
  - A. Every sequence with cluster point is bounded.
  - B. If  $\limsup x_n = 2$  then  $x_n \leq 2$  for  $n$  sufficiently large.
  - C. If  $\limsup x_n = 2$  then  $x_n > 1.99$  for  $n$  sufficiently large.
  - D. If  $\limsup x_n = 2$  then  $x_n < 2.01$  for  $n$  sufficiently large.
  - E. If  $\limsup x_n = 2$  and  $\liminf x_n = 1$  then  $\{x_n\}$  is bounded.
  - F. If  $x_n < 2$  for all  $n$  and  $\{x_n\}$  has a cluster point  $c$  then  $c < 2$ .
  - G. If  $\{x_n\}$  is a bounded and has only one cluster point  $c$  then  $x_n \rightarrow c$  as  $n \rightarrow \infty$ .
  - H. If  $\{x_n\}$  is a bounded sequence with only two cluster points  $a$  and  $b$  with  $a < b$  then  $a = \liminf x_n$  and  $b = \limsup x_n$ .
5. Prove that if  $f$  is a continuous real valued one to one function on the interval  $[a,b]$  and  $f(a) < f(b)$  then if  $c \in (a,b)$  then  $f(a) < f(c) < f(b)$ . (Hint. Suppose  $f(c) \geq f(b)$  or  $f(c) \leq f(a)$ ).
6. Prove that if  $f$  is differentiable on the open interval  $(a,b)$  and  $f'(x) = 0$  for  $x \in (a,b)$  then  $f(x) = C$  (a constant).
7. Prove that if  $f$  is differentiable in the open interval  $(a,b)$  and  $f'(x) > 0$  for  $x \in (a,b)$  then  $f$  is strictly increasing.
8. Show that if  $f$  is defined and continuous in the closed interval  $[a,b]$  and  $f$  is twice differentiable in the open interval  $(a,b)$  and  $f''(x) > 0$  for  $x \in (a,b)$  then if  $c \in (a,b)$  the point  $(c, f(c))$  lies below the straight line through  $(a, f(a))$  and  $(b, f(b))$ . (This means  $f$  is strictly convex.)
9. Suppose  $p(x) = x^3 - 3ax + b$ . Find the conditions on  $a$  and  $b$  so that  $p$  has three distinct roots.
10. Show that if  $q(x) = x^4 + ax^3 + bx^2 + cx + d$  has four distinct roots then  $q'(x)$  must have three distinct roots.
11. Suppose  $f$  is defined for all real  $x$  and  $|f(x) - f(y)| \leq 5(x - y)^2$  for all real  $x$  and  $y$ .
  - a) Prove  $f$  is continuous.
  - b) Prove  $f$  is differentiable with derivative  $f'(x) = 0$ .
  - c) Prove  $f$  is constant  $f(x) = C$  for all  $x$ .