Math 240: Spring-mass Systems

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Spring/Mass Systems with Damped Motion

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Review for Last Time

- Learned how to solve Cauchy-Euler Equations.
- Learned how to model spring/mass systems with undamped motion.

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Cauchy-Euler Equations

Definition

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is a Cauchy-Euler equation.

Note: These are **not** constant coefficient.

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Review

Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if m is a root of multiplicity k, then

$$x^{m}, x^{m} ln(x), \dots, x^{m} (ln(x))^{k-1}$$

are k linearly independent solutions

Undamped Spring-Mass Systems

By Newton's Second Law and Hooke's Law, the following D.E. models an undamped mass-spring system

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$$m\frac{d^2x}{dt^2} = -kx$$

where k is the spring constant, m is the mass placed at the end of the spring and x(t) is the position of the mass at time t.

Undamped Spring-Mass Systems

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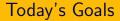
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where k is the spring constant, m is the mass placed at the end of the spring and x(t) is the position of the mass at time t.

Example: A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.

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Learn how to model spring/mass systems with damped motion.
Learn how to model spring/mass systems with driven motion.

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Spring/Mass Systems with Damped Motion

Undamped motion is unrealistic. Instead assume we have a damping force proportional to the instantaneous velocity.

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Spring/Mass Systems with Damped Motion

Undamped motion is unrealistic. Instead assume we have a damping force proportional to the instantaneous velocity.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

is now our model, where *m* is the mass, *k* is the spring constant, β is the damping constant and x(t) is the position of the mass at time *t*.

Changing Variables

Let

$$2\lambda = \frac{\beta}{m}$$
 and $\omega^2 = \frac{k}{m}$.

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and the roots of the Aux. Equation become

$$m_1=-\lambda+\sqrt{\lambda^2-\omega^2}$$
 and $m_2=-\lambda-\sqrt{\lambda^2-\omega^2}$

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Case 1: Overdamped

If $\lambda^2 - \omega^2 > 0$ the system is **overdamped** since β is large when compared to k. In this case the solution is

$$x=e^{-\lambda t}(c_1e^{\sqrt{\lambda^2-\omega^2}t}+c_2e^{\sqrt{\lambda^2-\omega^2}t}).$$

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Case 2: Critically Damped

If $\lambda^2 - \omega^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$x = e^{-\lambda t} (c_1 + c_2 t)$$

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Case 3: Underdamped

If $\lambda^2 - \omega^2 < 0$ the system is **underdamped** since k is large when compared to β . In this case the solution is

$$x = e^{-\lambda t}(c_1 cos(\sqrt{\lambda^2 - \omega^2}t) + c_2 sin(\sqrt{\lambda^2 - \omega^2}t))$$

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A 4 foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/sec.

Driven Motion

When an external force f(t) acts on the mass on a spring, the equation for our model of motion becomes

$$\frac{d^2x}{dt^2} = -\frac{\beta}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{f(t)}{m}$$

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or in the language of λ and $\omega,$

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$$

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When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest at equilibrium position. Starting at t = 0 a force of $f(t) = 65e^{-2t}$ is applied to the system. In the absence of damping, find the equation of motion.

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What is the amplitude of the oscillation after a very long time?

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