MATH 240 - Fall 2012
Practice Midterm Two

Name:

TA:

Recitation number:

You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. Show all work, even on multiple choice or short answer questions—I will be grading as much on the basis of work shown as on the end result. Remember to put your name at the top of this page. Good luck.

Problem	**Score (out of)**
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
Total	

1.(10 pt) Please mark "T" for true and "F" for false in the space provided to the left of the following statements. If the statement is false, **YOU MUST PROVIDE A COUNTEREXAMPLE FOR FULL CREDIT.**

T If A is an $n \times n$ matrix such that all of the eigenvalues of A are purely imaginary, then n is even.

T The vector space of all $n \times n$ skew-symmetric matrices has dimension $n(n-1)/2$.

F If zero is an eigenvalue of an $n \times n$ matrix A, then A is invertible.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalue 0 and corresponding eigen vectors $\langle 1, 0 \rangle, \langle 0, 1 \rangle$.

F If A is an $n \times n$ matrix, then $rowspace(A) = colspace(A)$.

If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $rowspace(A) = span\{\langle 0, 1 \rangle\}$ and $colspace(A) = span\{\langle 1, 0 \rangle\}$.

F The mapping $T : M_n(\mathbb{R}) \to \mathbb{R}$ given by $T(A) = det(A)$ is a linear transformation.

$T(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = T(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = det(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 4$

$2\ T(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 2 \cdot det(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 2 \cdot 1 = 2$

$2 \neq 4$

2. Derive the formula for the determinant of a 3×3 matrix by using the definition of determinant.

Example 3.1.9 in DELA

3. Find all of the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Step 1: Find eigenvalues: Solve $\det(A-\lambda I)=0$

$$\begin{vmatrix} -\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$-\lambda\big((-\lambda)(-1-\lambda)-0\big) - 1\big(1-(-\lambda)\cdot 1\big) = 0$$

$$-\lambda^2-\lambda^3-1-\lambda=0$$

$$\lambda^3+\lambda^2+\lambda+1=0 \leftarrow \text{time for long division}$$

long division

$$\begin{array}{r} \lambda^2 \qquad\quad +1 \\ \lambda+1 \;\big)\; \lambda^3+\lambda^2+\lambda+1 \\ -(\lambda^3+\lambda^2) \qquad\quad \\ \hline 0 \;+\lambda+1 \\ \lambda+1 \\ \hline 0 \end{array}$$

$$(\lambda+1)(\lambda^2+1)=0$$

$$\lambda = -1,\ i,\ -i$$

Step 2: Find eigen vectors: Solve $(A-\lambda I)v=0$

$\lambda=-1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{A_{23}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{A_{12}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x-z=0$$
$$y+z=0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

eigenvector!

$\lambda=i$

$$\left[\begin{array}{ccc|c} -i & 0 & -1 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 1 & -1-i & 0 \end{array}\right]$$

$A_{23}(-1)$ & $A_{21}(i)$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 1+i & -1-i & 0 \end{array}\right]$$

$A_{13}(-(1+i))$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$y-z=0$$
$$x-iy=0 \Rightarrow x-iz=0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} iz \\ z \\ z \end{bmatrix} = z\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}$$

eigenvec

$\lambda=-i$

Here we know the eigenvector will be the complex conjugate of the eigenvector for $\lambda=i$, so

$$\begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$$

eigenvec

4. Using the standard definitions of function addition and scalar times function, show that the set of all linear transformations from a vector space V to a vector space W is itself a vector subspace of the vector space of all continuous mappings from V to W.

① Show closed under vector addition.

Let T and ~~a~~ R be linear transformations from V to W.

Look at $(T+R)(v) = T(v) + R(v)$.

Must show $(T+R)(v)$ is a linear transformation from V to W.

1a) Show $(T+R)(v_1+v_2) = (T+R)(v_1) + (T+R)(v_2)$.

Let $v_1, v_2 \in V$.

Look at $(T+R)(v_1+v_2) = T(v_1+v_2) + R(v_1+v_2)$

Since T and R are both Linear Transformations,

$(T+R)(v_1+v_2) = T(v_1) + T(v_2) + R(v_1) + R(v_2)$.

By def of ~~T and~~ $T+R$

$(T+R)(v_1+v_2) = (T+R)(v_1) + (T+R)(v_2)$. $\square$

1b) Show $(T+R)(cv_1) = c(T+R)(v_1)$.

Let $c \in \mathbb{R}$ and let $v_1 \in V$.

Look at $(T+R)(cv_1) = T(cv_1) + R(cv_1)$.

Since T and R are Linear transformations,

$$(T+R)(cv_1) = c\,T(v_1) + c\,R(v_1)$$
$$= c\,(T(v_1) + R(v_1))$$
$$= c\,(T+R)(v_1). \ \square$$

Hence $T+R$ is a linear transformation.

② Show closed under scalar Multiplication.

Let T be a linear transformation from V to W and $c \in \mathbb{R}$.

Must show $cT(v)$ is a linear transformation.

2a) Let $v_1, v_2 \in V$. Look at $cT(v_1+v_2) \neq$ ~~[illegible]~~

$cT(v_1+v_2) \underset{\text{Since } T \text{ is a L.T.}}{=} c\,(T(v_1) + T(v_2)) = cT(v_1) + cT(v_2)$. $\square$

2b) Let $k \in \mathbb{R}$ and $v_1 \in V$.

Look at $cT(kv_1) \underset{\text{Since } T \text{ is a L.T.}}{=} c(kT(v_1)) = k(cT(v_1))$. $\square$

Thus $cT(v)$ is a linear transformation.

Hence, this set is a vector subspace. $\square$

5. Find a basis for the subspace of $M_2(\mathbb{R})$ consisting of all 2×2 matrices of trace zero. Prove that the set you find is a basis.

Claim: $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is a basis.

Step 1: Show these vectors span

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a trace zero matrix. Hence, $a+d=0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Hence, these vectors span.

Step 2: Show these vectors are linearly independant.

Look at $c_1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} c_3 & c_1 \\ c_2 & -c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $c_3 = 0$, $c_1 = 0$, $c_2 = 0$.

Thus, by definition of Linear Independence, the vectors $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are linearly independant.