Final

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RULES:

- You will be given the entire period (1PM-3:10PM) to complete the test.
- You can use one sheet of paper for formulas. There are no calculators nor those fancy cellular phones nor groupwork allowed. Each problem is worth 10 points, and partial credit is awarded.
- Show all of your work. Correct answers without sufficient work will be worth nearly nothing. Also the more work you show, the easier it is for me to find your mistakes and possibly give you more points.
- Be clear what it is that you want graded - if there a multiple solutions I will grade the first one.
- Good luck!!!

QUESTION 1	R / 10
QUESTION 2	Y / 10
QUESTION 3	A / 10
QUESTION 4	N / 10
QUESTION 5	M / 10
QUESTION 6	A / 10
QUESTION 7	N / 10
QUESTION 8	I / 10
QUESTION 9	O / 10
TOTAL	N / 90

#1. Find the rank and determinant of the following matrix. Also state whether or not it has an inverse.

$$A = \begin{pmatrix} -3 & 4 & 0 \\ -3 & 2 & 3 \\ 1 & -2 & 1 \end{pmatrix}$$

RANK: $R_1 \leftrightarrow R_3$ $\begin{pmatrix} 1 & -2 & 1 \\ -3 & 2 & 3 \\ -3 & 4 & 0 \end{pmatrix}$ $\begin{matrix} R_2 \longrightarrow R_2 + 3R_1 \\ R_3 \longrightarrow R_3 + 3R_1 \end{matrix}$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -4 & 6 \\ 0 & -2 & 3 \end{pmatrix} \quad R_3 \longrightarrow R_3 - \tfrac{1}{2}R_2$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -4 & 6 \\ 0 & 0 & 0 \end{pmatrix} \qquad RkA = 2$$

THUS SINCE $RkA = 3 \iff \det A \neq 0 \iff A^{-1}$ EXISTS, WE HAVE

$\boxed{\det A = 0 \text{ AND } A^{-1} \text{ DOES NOT EXIST}}$

#2. Find $\det((AB)^T A^{-1})$ if we have:

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 5 & 7 & 2 \end{pmatrix}$$

$$\begin{aligned}
\text{DET}\left((AB)^T A^{-1}\right) &= \text{DET}\left(B^T A^T A^{-1}\right) \\
&= \text{DET}(B^T)\,\text{DET}(A^T)\,\text{DET}(A^{-1}) \\
&= \text{DET}(B)\,\underbrace{\text{DET}(A)\,\text{DET}(A^{-1})}_{=1} \\
&= \text{DET}(B) \\
&= \begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 5 & 7 & 2 \end{vmatrix} \\
&= 2\begin{vmatrix} 2 & 3 \\ 7 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 5 & 7 \end{vmatrix} \\
&= 2(4-21) + (-7-10) \\
&= -34 - 17 \\
&= -51
\end{aligned}$$

#3. For what s and t are the following three vectors linearly independent?

$$\begin{pmatrix} 3 \\ s \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ t+3 \\ t \end{pmatrix}$$

VECTORS ARE L.I. $\iff A = \begin{pmatrix} 3 & 0 & 1 \\ s & -1 & t+3 \\ -2 & 4 & t \end{pmatrix}$ HAS RANK 3

$\iff A^{-1}$ EXISTS

$\iff \det A \neq 0$

$$\det A = \begin{vmatrix} 3 & 0 & 1 \\ s & -1 & t+3 \\ -2 & 4 & t \end{vmatrix}$$

$$= 3 \begin{vmatrix} -1 & t+3 \\ 4 & t \end{vmatrix} + \begin{vmatrix} s & -1 \\ -2 & 4 \end{vmatrix}$$

$$= 3(-t - 4t - 12) + (4s - 2)$$

$$= -15t - 36 + 4s - 2$$

$$= -15t + 4s - 38$$

AS LONG AS $\det A = -15t + 4s - 38 \neq 0$

THESE VECTORS ARE L. I.

(ANOTHER METHOD WOULD BE TO ROW REDUCE A AND FIND FOR WHAT s & t A HAS RANK 3)

#4. Find the general solution to the following differential equation:

$$2y^{(4)} + y''' - 6y'' = e^{2x}$$

Y_H: $\quad Y = e^{mx}$: $\quad 2m^4 + m^3 - 6m^2 = 0$

$$\underbrace{m^2}_{m=0,0}(\underbrace{2m + m - 6}) = 0$$

$$m = \frac{1}{4}\left(-1 \pm \sqrt{1+48}\right)$$

$$m = \frac{1}{4}(-1 \pm 7)$$

$$m = -2, \frac{3}{2}$$

$$Y_H = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{\frac{3}{2}x}$$

Y_P: GUESS $Y_P = Ae^{2x}$ $\qquad Y_P''' = 8Ae^{2x}$

$Y_P' = 2Ae^{2x}$ $\qquad Y_P^{(4)} = 16Ae^{2x}$ $\Big\}$ PLUG IN:

$Y_P'' = 4Ae^{2x}$

$$32Ae^{2x} + 8Ae^{2x} - 24Ae^{2x} = e^{2x}$$

$$16Ae^{2x} = e^{2x}$$

$$A = \frac{1}{16}$$

$Y = Y_H + Y_P$

$$Y = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{\frac{3}{2}x} + \frac{1}{16}e^{2x}$$

#5. Find the general solution to the system of differential equations:

$$X' = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix} X$$

FIND λ's: $\begin{vmatrix} -7-\lambda & -6 \\ 9 & 8-\lambda \end{vmatrix} = (-7-\lambda)(8-\lambda) + 54$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1)$$

$$\lambda_1 = -1 \quad \lambda_2 = 2$$

V_1: $(A - (-1)I)\, V_1 = 0$

$$\left(\begin{array}{cc|c} -6 & -6 & 0 \\ 9 & 9 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

V_2: $(A - 2I)\, V_2 = 0$

$$\left(\begin{array}{cc|c} -9 & -6 & 0 \\ 9 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$X = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

#6. Find the series solution about the point $x = 0$ to the following differential equation up to the x^5 term and satisfying the initial conditions $y(0) = 2$ and $y'(0) = 1$:

$$(x^2 - 1)y'' - 2y = 0$$

X = 0 NOT SINGULAR, USE $y = \sum_{n\geq 0} c_n x^n$ $\qquad y(0) = c_0 = 2$

$y'(0) = c_1 = 1$

$$(x^2-1)\sum_{n\geq 2} n(n-1)c_n x^{n-2} - 2\sum_{n\geq 0} c_n x^n = 0$$

$$\underbrace{\sum_{n\geq 2} n(n-1)c_n x^n}_{k=n} - \underbrace{\sum_{n\geq 2} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n\geq 0} 2c_n x^n}_{k=n} = 0$$

$$\sum_{k\geq 2} k(k-1)c_k x^k - \underbrace{\sum_{k\geq 0}(k+2)(k+1)c_{k+2}x^k - \sum_{k\geq 0} 2c_k x^k}_{\text{PULL OUT } k=0,1 \text{ TERMS \& COMBINE}} = 0$$

$$(-2c_2 - 2c_0)x^0 + (-6c_3 - 2c_1)x^1 + \sum_{k\geq 2}\left[-(k+2)(k+1)c_{k+2} + \underbrace{(k^2-k-2)}_{(k-2)(k+1)}c_k\right]x^k = 0$$

$-2c_2 = 2c_0$

$c_2 = -c_0$

$c_2 = -2$

$-6c_3 = 2c_1$

$c_3 = -\frac{1}{3}c_1$

$c_3 = -\frac{1}{3}$

RECURRENCE:

$$c_{k+2} = \frac{(k-2)(k+1)}{(k+2)(k+1)}c_k = \frac{k-2}{k+2}c_k$$

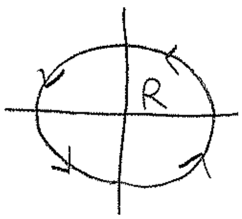
$k=2$: $c_4 = 0c_2 = 0$ $\qquad k=3$: $c_5 = \frac{1}{5}c_3 = -\frac{1}{15}$

THUS OUR SOLUTION IS

$$y = 2 + x - 2x^2 - \frac{1}{3}x^3 - \frac{1}{15}x^5 + \dots$$

#7. Compute the following line integral around the circle radius 2 centered at the origin in the counterclockwise direction:

$$\oint \underbrace{(x^3 - x^2y^3)}_{P}dx + \underbrace{(x^3y^2 + y^3 - y^2)}_{Q}dy$$



$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$$

$$= \iint 3x^2y^2 - (-3x^2y^2) \, dx\,dy$$

$$= 6 \iint x^2y^2 \, dx\,dy \longrightarrow \underline{\text{POLAR}}$$

$$= 6 \int_0^{2\pi}\int_0^{2} r^5 \cos^2\theta \sin^2\theta \, dr\, d\theta$$

$$= \cancel{6} \int \cancel{\tfrac{1}{6}} r^6 \Big|_0^2 \cos^2\theta \sin^2\theta \, d\theta \longrightarrow \text{USE TRIG IDENTITIES}$$

$$= 64 \int \left(\tfrac{1}{2}\right)(1 + \cos 2\theta)\left(\tfrac{1}{2}\right)(1 - \cos 2\theta)\, d\theta$$

$$= 16 \int 1 - \cos^2 2\theta \, d\theta$$

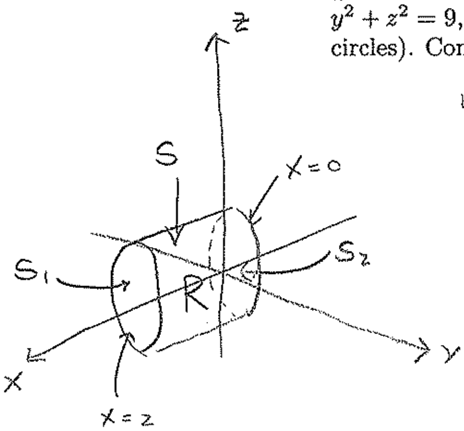
$$= 16 \int 1 - \left(\tfrac{1}{2}\right)(1 + \cos 4\theta)\, d\theta$$

$$= 16 \int \tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta \, d\theta$$

$$= 16 \left[\tfrac{1}{2}\theta - \cancel{\tfrac{1}{8}\sin 4\theta} \right] \Bigg|_0^{2\pi}$$

$$= 16\pi$$

#8. Let S be the surface around the round sides of the cylinder cut out by $y^2+z^2=9$, $x=0$, and $x=2$ (meaning not including the flat top and bottom circles). Compute the flux integral $\int\int_S F\cdot ndS$ if $F=(x^2, y+xe^z, \cos xy)$



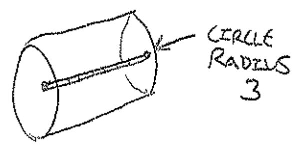
LABEL THE TOP & BOTTOM S_1 & S_2 AS IN THE PIC.

DIV THM: (R IS OUR SOLID CYLINDER)

$$\underbrace{\iiint_R \operatorname{div} F\, dV}_{\text{I}} = \underbrace{\iint_S F\cdot n\, dS}_{\text{WHAT WE WANT}} + \underbrace{\iint_{S_1} F\cdot n\, dS}_{\text{II}} + \underbrace{\iint_{S_2} F\cdot n\, dS}_{\text{III}}$$

VOLUME OF CYLINDER

$$\text{I} = \iiint_R 2x+1\, dV = 2\iiint x\, dV + \iiint dV$$



$$= 2\iint\int_0^2 x\, dx\, dy\, dz + 18\pi$$

$$= 2\iint \tfrac{1}{2}x^2\Big|_0^2\, dy\, dz + 18\pi$$

$$= 4\underbrace{\iint dy\, dz}_{\text{OVER CIRCLE RADIUS 3}} + 18\pi$$

$$= 4(9\pi) + 18\pi$$

$$= 54\pi$$

X IS CONSTANT ON S_1

II HERE $n=(1,0,0)$ so $\iint_{S_1} F\cdot n\, dS = \iint x^2\, dS = \iint 4\, dS = 4$ (AREA OF CIRCLE)

$$= 4(9\pi) = 36\pi$$

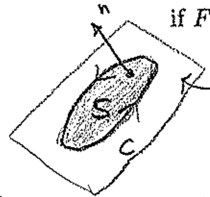
III HERE $n=(-1,0,0)$ so $\iint_{S_2} F\cdot n\, dS = -\iint x^2\, dS = 0$ ($x=0$ ON S_2)

$$\iint_S F\cdot n\, dS = \text{I} - \text{II} - \text{III}$$

$$= 54\pi - 36\pi - 0$$

$$= 18\pi$$

#9. Let C be the curve formed by the intersection of the plane $z = x + 2y + 10$ and the cylinder $x^2 + y^2 = 1$ traversed in a counterclockwise direction when looking down from the positive z-axis. Compute the circulation integral $\oint_C F \cdot dr$ if $F = (ze^x, \frac{1}{3}x^3, -\frac{1}{3}y^3 + e^x)$.



$Z = X + 2Y + 10$

$$\oint_C F \cdot dr = \iint_S \text{CURL} F \cdot n \, dS \qquad \text{STOKES'}$$

$$\text{CURL} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & \frac{1}{3}x^3 & -\frac{1}{3}y^3 + e^x \end{vmatrix}$$

$$= \left(-y^2, -[e^x - e^x], x^2\right)$$

$$= (-y^2, 0, x^2)$$

$$\sigma = (x, y, x + 2y + 10)$$

$$\sigma_x = (1, 0, 1)$$

$$\sigma_y = (0, 1, 2)$$

$$\sigma_x \times \sigma_y = (-1, -2, 1)$$

CORRECT SINCE WE WANT NORMAL W/ + Z-COORD (SEE PIC)

$$\oint_C F \cdot dr = \iint_S \text{CURL} F \cdot n \, dS = \iint (-y^2, 0, x^2) \cdot \overbrace{\frac{(-1, -2, 1)}{\cancel{\sqrt{6}}}}^{n} \overbrace{\cancel{\sqrt{6}} \, dx \, dy}^{dS}$$

$$= \iint x^2 + y^2 \, dx \, dy \qquad \text{(INTEGRATING OVER THE UNIT CIRCLE)}$$

$$= \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta$$

$$= \frac{1}{4} r^4 \Big|_0^1 \int d\theta$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

#E. (EXTRA CREDIT) This problem is worth NO POINTS. You must finish the rest of the exam before you attempt it. It will be used as an extra consideration when assigning your final overall grades.

a. Show that the usual dot product $v \cdot w$ of vectors $v, w \in \mathbb{R}^n$ can be expressed as a product of matrices, thinking of v and w as $n \times 1$ matrices. Hint: Use a matrix transpose!
b. Show that the length $||v||$ of a vector v can be expressed in terms of a dot product (and thus can be written using matrix products by above.)
c. Recall that an orthogonal matrix A is a matrix such that $A^T = A^{-1}$. Show that such an A preserves the lengths of vectors and the angles between two vectors (use the previous two parts.) More precisely show $||Av|| = ||v||$ and the angle between Av and Aw is the same as the angle between v and w, or equivalently that the cosines of these angles are equal. (You might want to show the lengths first.)

a. $V \cdot W = \underset{1\times n}{V^T}\,\underset{n \times 1}{W}$ SO $V^T W = 1 \times 1$ (I.E. A NUMBER) IS THE DOT PRODUCT

b. $V \cdot V = \|V\|\,\|V\| \underbrace{\cos\theta}_{\theta = 0 \text{ IN THIS CASE}} = \|V\|^2$ SO $\|V\| = \sqrt{V\cdot V} = \sqrt{V^T V}$ BY (a)

c. LENGTHS:

$$\|Av\| \underset{\text{BY ABOVE}}{=} \sqrt{(Av)\cdot(Av)} \underset{\text{BY ABOVE}}{=} \sqrt{(Av)^T Av} \underset{\text{DISTRIB. T}}{=} \sqrt{v^T A^T A v} \underset{A \text{ ORTHOG.}}{=} \sqrt{v^T \underbrace{A^{-1} A}_{\text{CANCEL}} v} = \sqrt{v^T v} = \sqrt{v \cdot v} = \|v\|$$

ANGLES:

$$\left.\begin{aligned} Av \cdot Aw &= \|Av\|\,\|Aw\| \cos\theta_1 \\ v\cdot w &= \|v\|\,\|w\| \cos\theta_2 \end{aligned}\right\} \text{ WE WANT } \theta_1 = \theta_2$$

$$Av \cdot Aw = (Av)^T Aw = v^T A^T A w = v^T A^{-1} A w = v^T w = v \cdot w$$

SO THE DOT PRODUCTS ARE EQUAL AND THUS

$$\|Av\|\,\|Aw\| \cos\theta_1 = \|v\|\,\|w\| \cos\theta_2$$

$\downarrow$ A PRESERVES LENGTHS

$$\cancel{\|v\|\,\|w\|} \cos\theta_1 = \cancel{\|v\|\,\|w\|} \cos\theta_2$$

$$\cos\theta_1 = \cos\theta_2$$

$$\theta_1 = \theta_2$$