① $Ax = b$ HAS A SOLUTION FOR ANY b IFF A HAS RANK 3. SO LET'S COMPUTE RK A

$$\begin{pmatrix} 7 & 5 & 9 \\ 1 & -1 & 3 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 3 \\ 7 & 5 & 9 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 7R_1} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 12 & -12 \\ 0 & 6 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{RK} A = 2$

THUS $Ax = b$ DOES NOT HAVE A SOLUTION FOR ANY b

(THE SET OF ALL VECTORS OF THE FORM Ax IS A PLANE SINCE $\text{RK} A = 2$)

② A^{-1} EXISTS IFF $\det A \neq 0$ SO:

$$\det A = \begin{vmatrix} a & -1 & 0 \\ b & 2 & -2 \\ 3-b & 0 & 4 \end{vmatrix} = a \begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} b & -2 \\ 3-b & 4 \end{vmatrix}$$

$$= 8a + [4b + 2(3-b)]$$

$$= 8a + 2b + 6$$

AS LONG AS $8a + 2b + 6 \neq 0$

OR $4a + b + 3 \neq 0$

③ USE ROW OPS

FIRST DO

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 2R_1$

$$\text{DET} = \begin{vmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 & -10 \\ 0 & -3 & -1 & 3 & -4 \\ 0 & 1 & -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 5 \\ 0 & 1 & 4 & -10 \\ -3 & -1 & 3 & -4 \\ 1 & -1 & 1 & 3 \end{vmatrix} \quad R_3 \to R_3 + 3R_4$$

$$= \begin{vmatrix} 0 & 1 & 2 & 5 \\ 0 & 1 & 4 & -10 \\ 0 & -4 & 6 & 5 \\ 1 & -1 & 1 & 3 \end{vmatrix}$$

EXPANDING AT 1 GIVES YOU A NEGATIVE!

$$= -\begin{vmatrix} 1 & 2 & 5 \\ 1 & 4 & -10 \\ -4 & 6 & 5 \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 4R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 2 & -15 \\ 0 & 14 & 25 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & -15 \\ 14 & 25 \end{vmatrix}$$

$$= -260$$

(4) DISTINCT λ'S IMPLIES DIAGONALIZABLE

$$\begin{vmatrix} 1-\lambda & 1+c \\ 1-c & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (1-c^2) = 0$$

$$\lambda^2 - 2\lambda + 1 - 1 + c^2 = 0$$

$$\lambda^2 - 2\lambda + c^2 = 0$$

$$\lambda = \frac{1}{2}\left(2 \pm \sqrt{4-4c^2}\right)$$

$$\lambda = 1 \pm \sqrt{1-c^2}$$

$1-c^2 \neq 0 \implies \lambda$'S DISTINCT $\implies$ DIAGONALIZABLE

$1-c^2 = 0 \implies c = \pm 1$

IF $c=1$: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ IS OUR MATRIX, $\lambda_1 = \lambda_2 = 1$

AND ONLY <u>ONE</u> L.I. E.VECTOR:

$$(A - 1I)v = 0 \longrightarrow \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{SO} \quad \begin{matrix} 2y = 0 \\ y = 0 \end{matrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ ONLY L.I. E. VECTOR}$$

SO $c \neq 1$

IF $c = -1$: $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ IS OUR MATRIX, $\lambda_1 = \lambda_2 = 1$ & SAME THING HAPPENS

SO AS LONG AS $c \neq \pm 1$ IT IS DIAGONALIZABLE

(5) $\#\text{PARAMS} = \#\text{VARS (COLUMNS OF } A) - \text{RK}(A)$

$= 5 - \text{RK}(A)$

SO COMPUTE RK A:

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \quad \begin{pmatrix} 1 & 3 & 5 & -1 & 0 \\ 0 & -1 & -2 & 2 & 1 \\ 0 & -10 & -19 & 3 & 0 \end{pmatrix} \quad R_3 \to R_3 - 10R_2 \quad \begin{pmatrix} 1 & 3 & 5 & -1 & 0 \\ 0 & -1 & -2 & 2 & 1 \\ 0 & 0 & 1 & -17 & -10 \end{pmatrix}$$

RK A = 3

#PARAMS = 2

(6) $y'' - 2y' + 2y = e^x + 5$

y_H: $m^2 - 2m + 2 = 0$

$m = \frac{1}{2}\left(2 \pm \sqrt{4-8}\right)$

$m = 1 \pm i$

$y_H = c_1 e^x \cos x + c_2 e^x \sin x$

y_p: GUESS $y_p = Ae^x + B$

$y_p' = Ae^x$

$y_p'' = Ae^x$

$\Longrightarrow$ $Ae^x - 2Ae^x + 2(Ae^x + B) = e^x + 5$

$Ae^x + 2B = e^x + 5$

so $A = 1$

$2B = 5 \longrightarrow B = \frac{5}{2}$

$y = y_H + y_p = c_1 e^x \cos x + c_2 e^x \sin x + e^x + \frac{5}{2}$

(7) $y'' + 2y' - 3y = 2e^{-3x}$

y_H: $m^2 + 2m - 3 = 0$

$(m+3)(m-1) = 0$

$m = 1, -3$

$y_H = c_1 e^x + c_2 e^{-3x}$

y_p: $y_p = Axe^{-3x}$

$y_p' = A(1-3x)e^{-3x}$

$y_p'' = A\left[-3e^{-3x} + (1-3x)(-3)e^{-3x}\right] = A\left[-6 + 9x\right]e^{-3x}$

$A(-6+9x)e^{-3x} + 2A(1-3x)e^{-3x} - 3Axe^{-3x} = 2e^{-3x}$

$(-6A + 2A)e^{-3x} + (\cancel{9A - 6A - 3A})xe^{-3x} = 2e^{-3x}$

$-4Ae^{-3x} = 2e^{-3x}$

$A = -\frac{1}{2}$ so $y = c_1 e^x + c_2 e^{-3x} - \frac{1}{2}xe^{-3x}$

(8) $x^2y'' + 2xy' - 4y = 0 \qquad y(1) = 0, \ y'(1) = 1$

CAUCHY EULER $\quad y = x^m$ PLUG IN:

$$\left(m(m-1) - 2m - 4 \right) x^m = 0$$

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$m = -1, 4$$

$$y = \frac{c_1}{x} + c_2 x^4 \quad \text{GEN. SOL.}$$

$$y(1) = c_1 + c_2 = 0 \quad \text{so} \quad c_2 = -c_1$$

$$y' = -\frac{c_1}{x^2} + 4c_2x^3$$

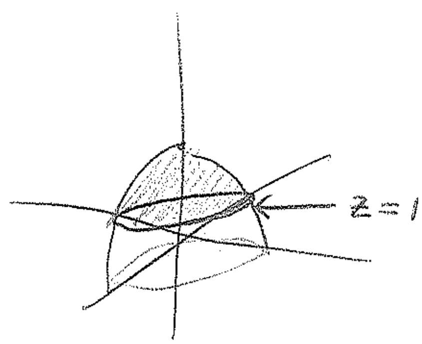
$$y'(1) = -c_1 + 4c_2 = 1$$

$$-5c_1 = 1$$

$$c_1 = -\frac{1}{5}, \quad c_2 = \frac{1}{5}$$

$$y = -\frac{1}{5x} + \frac{1}{5}x^4$$

(9) $z = 4 - x^2 - y^2 \qquad z \geq 1$



$$\sigma = (x, y, 4 - x^2 - y^2)$$

$$\sigma_x = (1, 0, -2x)$$

$$\sigma_y = (0, 1, -2y)$$

$$\sigma_x \times \sigma_y = (2x, 2y, 1)$$

$\rightarrow$ $z = 1$ THEN $x^2 + y^2 = 3$

CIRCLE RADIUS $\sqrt{3}$ IS OUR "SHADOW"

$$SA = \iint dS = \iint \|\sigma_x \times \sigma_y\| \, dx\,dy$$

$$= \iint \sqrt{4x^2 + 4y^2 + 1} \, dx\,dy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2+1} \, r \, dr\,d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$$= \frac{1}{8} \iint u^{\frac{1}{2}} \, du \, d\theta$$

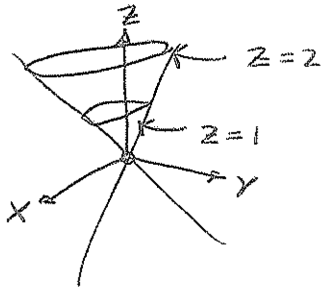
$$= \frac{1}{8} \int \frac{2}{3} \left(4r^2 + 1 \right)^{\frac{3}{2}} \Big|_0^{\sqrt{3}} d\theta$$

$$\frac{1}{12} \int 13^{\frac{3}{2}} - 1 \, d\theta$$

$$\frac{13^{3/2} - 1}{12} (2\pi)$$

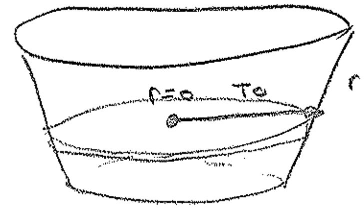
(10) $z^2 = x^2 + y^2 \quad 1 \le z \le 2$

CONE ↑



$$\iint_{\partial R} F \cdot n \, dS = \iiint_R \text{DIV} F \, dV$$

$$= \iiint x^2 + y^2 \, dV$$ CYLINDRICAL



r = CONE $z^2 = x^2 + y^2$

$z^2 = r^2$

$\pm z = r$

USE $r = z$

$(r > 0)$

$$= \int_1^2 \int_0^{2\pi} \int_0^z r^3 \, dr \, d\theta \, dz$$

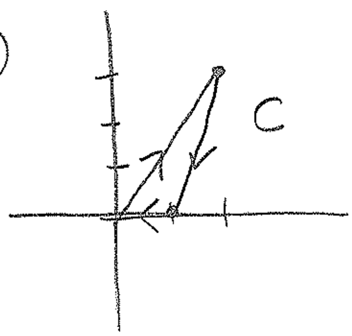
$$= \iint \tfrac{1}{4} z^4 \, d\theta \, dz$$

$$= \frac{\pi}{2} \int_1^2 z^4 \, dz$$

$$= \frac{\pi}{10} z^5 \Big|_1^2$$

$$= \frac{\pi}{10}(31)$$

(11)

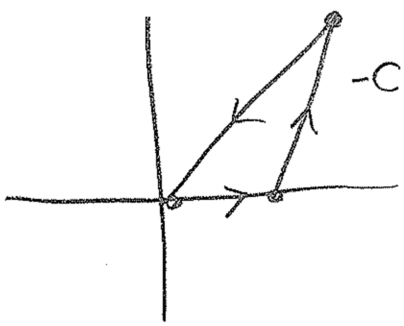


IN GREEN'S THM WE ALWAYS GO COUNTERCLOCKWISE!
SO WE MUST REVERSE OUR PATH AND GO ↺
THIS INTRODUCES A NEGATIVE.

$$\oint_C x \, dy = -\oint_{-C} x \, dy = -\oint_{-C} \underset{P}{0} \, dx + \underset{Q}{x} \, dy$$

$$= -\iint \underbrace{\frac{\partial Q}{\partial x}}_{1} - \underbrace{\frac{\partial P}{\partial y}}_{0} \, dx \, dy$$

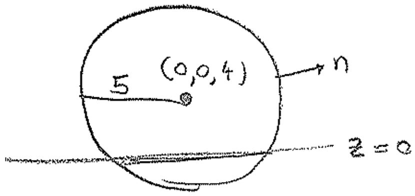
$$= -\iint dx \, dy$$

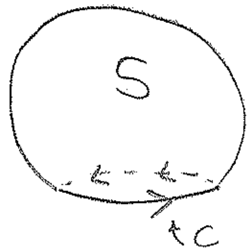


= -AREA OF Δ

$$= -\tfrac{1}{2} bh = -\tfrac{1}{2}(1)(3) = -\frac{3}{2}$$

(12)





C IS WHEN $z=0$:

$$x^2+y^2+(0-4)^2=25$$

$$x^2+y^2=9$$

RAD 3 CIRCLE

$$F=(y,\ y-x,\ z^2)$$

$$r(t)=(3\cos t,\ 3\sin t,\ 0)$$

$$\iint_S \text{curl}\,F\cdot n\,dS = \oint_C F\cdot dr$$

$$=\int_0^{2\pi} F\cdot r'(t)\,dt$$

$$=\int (3\sin t,\ 3(\sin t-\cos t),\ 0)\cdot(-3\sin t,\ 3\cos t,\ 0)\,dt$$

$$=\int -9\sin^2 t+9\cos t\sin t-9\cos^2 t\,dt$$

$$=9\int \cos t\sin t-1\,dt$$

$$=9\left[\tfrac{1}{2}\sin^2 t - t\right]\Big|_0^{2\pi}$$

$$=-18\pi$$

(13)

$$(x^2+1)\sum_{n\geq 2} n(n-1)c_n x^{n-2} - 6\sum_{n\geq 0} c_n x^n = 0 \quad (x=0 \text{ IS NOT SINGULAR})$$

$$\sum_{n\geq 2} n(n-1)c_n x^n + \sum_{n\geq 2} n(n-1)c_n x^{n-2} - \sum_{n\geq 0} 6c_n x^n = 0$$

$k=n$ $\quad\quad$ $k=n-2$ $\quad\quad$ $k=n$

$$\sum_{k\geq 2} k(k-1)c_k x^k + \underbrace{\sum_{k\geq 0}(k+2)(k+1)c_{k+2}x^k - \sum_{k\geq 0} 6c_k x^k}_{\text{PULL OUT } k=0,1 \text{ TERMS}} = 0$$

$$(2c_2 - 6c_0)x^0 + (6c_3 - 6c_1)x^1 + \dots$$

$$\dots \sum_{k\geq 2}\left[(k+2)(k+1)c_{k+2} + \underbrace{[k^2-k-6]}_{(k-3)(k+2)}c_k\right]x^k = 0$$

So $2c_2 = 6c_0$ $\qquad 6c_3 = 6c_1$

$c_2 = 3c_0$ $\qquad c_3 = c_1$

RECURRENCE: $c_{k+2} = \dfrac{-(k-3)\cancel{(k+2)}c_k}{\cancel{(k+2)}(k+1)} = \dfrac{(3-k)}{k+1}c_k$

$k=2$: $\quad c_4 = \frac{1}{3}\cdot c_2 = c_0$

$k=3$: $\quad c_5 = 0$

$k=4$: $\quad c_6 = \frac{-1}{5}c_4 = -\frac{1}{5}c_0$

$k=5$: $\quad c_7 = \frac{-4}{8}c_5 = 0$ (ALL ODD c's $= 0$)

$\vdots$

$$y = \underbrace{c_0\left(1 + 3x^2 - \tfrac{1}{5}x^4 + \dots\right)}_{y_1} + \underbrace{c_1\left(x + x^3\right)}_{y_2}$$

(14) $\quad X' = \begin{pmatrix} -7 & 12 \\ -4 & 7 \end{pmatrix}X \qquad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{vmatrix} -7-\lambda & 12 \\ -4 & 7-\lambda \end{vmatrix} = (-7-\lambda)(7-\lambda) + 48 = 0$$

$$-49 + \lambda^2 + 48 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$\lambda_1 = 1$:

$(A - I)v = 0$

$$\left(\begin{array}{cc|c} -8 & 12 & 0 \\ -4 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda_2 = -1$:

$(A + I)v = 0$

$$\left(\begin{array}{cc|c} -6 & 12 & 0 \\ -4 & 8 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

GEN SOL:

$$X = c_1 e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 3c_1 + 2c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $c_2 = -2c_1$, $\quad 3c_1 + 2c_2 = 1$

$$3c_1 - 4c_1 = 1$$

$c_1 = -1$ $\quad$ $c_2 = 2$

$$X = -e^{t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$X(1) = -e \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{2}{e} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$