for every **A** and **B** in $M_{m,n}$ and for every scalar k. That is to say, $M_{m,n}$ is closed under matrix addition and scalar multiplication. When we combine (5) with properties (3) and (4) and the properties listed in Theorem 8.1, it follows immediately that $M_{m,n}$ is a vector space. For practical purposes, the vector spaces $M_{1,n}$ (row vectors) and $M_{n,1}$ (column vectors) are indistinguishable from the vector space R^n.

EXERCISES 8.1

Answers to selected odd-numbered problems begin on page ANS-16.

In Problems 1–6, state the size of the given matrix.

1. $\begin{pmatrix} 1 & 2 & 3 & 9 \\ 5 & 6 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 2 \\ 8 & 4 \\ 5 & 6 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 7 & -2 \\ 0 & 0 & 5 \end{pmatrix}$

4. $(5 \quad 7 \quad -15)$

5. $\begin{pmatrix} 1 & 5 & -6 & 0 \\ 7 & -10 & 2 & 12 \\ 0 & 9 & 2 & -1 \end{pmatrix}$

6. $\begin{pmatrix} 1 \\ 5 \\ -6 \\ 0 \\ 7 \\ -10 \\ 2 \\ 12 \end{pmatrix}$

In Problems 7–10, determine whether the given matrices are equal.

7. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

9. $\begin{pmatrix} \sqrt{(-2)^2} & 1 \\ 2 & \frac{2}{8} \end{pmatrix}$, $\begin{pmatrix} -2 & 1 \\ 2 & \frac{1}{4} \end{pmatrix}$

10. $\begin{pmatrix} \frac{1}{8} & \frac{1}{5} \\ \sqrt{2} & 1 \end{pmatrix}$, $\begin{pmatrix} 0.125 & 0.2 \\ 1.414 & 1 \end{pmatrix}$

In Problems 11 and 12, determine the values of x and y for which the matrices are equal.

11. $\begin{pmatrix} 1 & x \\ y & -3 \end{pmatrix}$, $\begin{pmatrix} 1 & y-2 \\ 3x-2 & -3 \end{pmatrix}$

12. $\begin{pmatrix} x^2 & 1 \\ y & 5 \end{pmatrix}$, $\begin{pmatrix} 9 & 1 \\ 4x & 5 \end{pmatrix}$

In Problems 13 and 14, find the entries c_{23} and c_{12} for the matrix $\mathbf{C} = 2\mathbf{A} - 3\mathbf{B}$.

13. $\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 6 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -2 & 6 \\ 1 & 3 & -3 \end{pmatrix}$

14. $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 0 & -4 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 4 & 0 \\ 3 & 0 & 7 \end{pmatrix}$

15. If $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -6 & 9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 6 \\ 8 & -10 \end{pmatrix}$, find (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{B} - \mathbf{A}$, (c) $2\mathbf{A} + 3\mathbf{B}$.

16. If $\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 4 & 1 \\ 7 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -4 & -2 \end{pmatrix}$, find (a) $\mathbf{A} - \mathbf{B}$, (b) $\mathbf{B} - \mathbf{A}$, (c) $2(\mathbf{A} + \mathbf{B})$.

17. If $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 6 \\ 3 & 2 \end{pmatrix}$, find (a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $\mathbf{A}^2 = \mathbf{AA}$, (d) $\mathbf{B}^2 = \mathbf{BB}$.

18. If $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 5 & 10 \\ 8 & 12 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix}$, find (a) $\mathbf{AB}$, (b) $\mathbf{BA}$.

19. If $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}$, find (a) $\mathbf{BC}$, (b) $\mathbf{A}(\mathbf{BC})$, (c) $\mathbf{C}(\mathbf{BA})$, (d) $\mathbf{A}(\mathbf{B} + \mathbf{C})$.

20. If $\mathbf{A} = (5 \quad -6 \quad 7)$, $\mathbf{B} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$, find (a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $(\mathbf{BA})\mathbf{C}$, (d) $(\mathbf{AB})\mathbf{C}$.

21. If $\mathbf{A} = \begin{pmatrix} 4 \\ 8 \\ -10 \end{pmatrix}$ and $\mathbf{B} = (2 \quad 4 \quad 5)$, find (a) $\mathbf{A}^T\mathbf{A}$, (b) $\mathbf{B}^T\mathbf{B}$, (c) $\mathbf{A} + \mathbf{B}^T$.

22. If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 3 \\ 5 & 7 \end{pmatrix}$, find (a) $\mathbf{A} + \mathbf{B}^T$, (b) $2\mathbf{A}^T - \mathbf{B}^T$, (c) $\mathbf{A}^T(\mathbf{A} - \mathbf{B})$.

23. If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 8 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 10 \\ -2 & -5 \end{pmatrix}$, find (a) $(\mathbf{AB})^T$, (b) $\mathbf{B}^T\mathbf{A}^T$.

24. If $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 11 \\ -7 & 2 \end{pmatrix}$, find (a) $\mathbf{A}^T + \mathbf{B}$, (b) $2\mathbf{A} + \mathbf{B}^T$.

In Problems 25–28, write the given sum as a single column matrix.

25. $4\begin{pmatrix} -1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 8 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

26. $3\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

27. $\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 & 6 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -7 \\ 2 \end{pmatrix}$

28. $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 5 & -1 \\ 0 & -4 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ -6 \end{pmatrix}$

In Problems 29 and 30, determine the size of the matrix **A** such that the given product is defined.

29. $\begin{pmatrix} 2 & 1 & 3 & 3 \\ 9 & 6 & 7 & 0 \end{pmatrix}\mathbf{A}\begin{pmatrix} 0 \\ 5 \\ 7 \\ 9 \\ 2 \end{pmatrix}$

30. $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 9 & 6 \\ 7 & 0 & -1 \end{pmatrix}\mathbf{A}\begin{pmatrix} 0 & 1 \\ 7 & 4 \end{pmatrix}$

In Problems 31–34, suppose $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix}$. Verify the given property by computing the left and right members of the given equality.

31. $(\mathbf{A}^T)^T = \mathbf{A}$

32. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

33. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

34. $(6\mathbf{A})^T = 6\mathbf{A}^T$

35. Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 2 & 5 \end{pmatrix}$. Verify that the matrix $\mathbf{B} = \mathbf{AA}^T$ is symmetric.

36. Show that if **A** is an $m \times n$ matrix, then $\mathbf{AA}^T$ is symmetric.

37. In matrix theory, many of the familiar properties of the real number system are not valid. If a and b are real numbers, then $ab = 0$ implies that $a = 0$ or $b = 0$. Find two matrices such that $\mathbf{AB} = \mathbf{0}$ but $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$.

38. If a, b, and c are real numbers and $c \neq 0$, then $ac = bc$ implies $a = b$. For matrices, $\mathbf{AC} = \mathbf{BC}$, $\mathbf{C} \neq \mathbf{0}$, does not necessarily imply $\mathbf{A} = \mathbf{B}$. Verify this for

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 1 & 6 \\ 9 & 2 & -3 \\ -1 & 3 & 7 \end{pmatrix},$$

and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

In Problems 39 and 40, let **A** and **B** be $n \times n$ matrices. Explain why, in general, the given formula is not valid.

39. $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

40. $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$

41. Write $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ without matrices.

42. Write the system of equations

$$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9 \end{aligned}$$

as a matrix equation $\mathbf{AX} = \mathbf{B}$, where **X** and **B** are column vectors.

43. Verify that the quadratic form $ax^2 + bxy + cy^2$ is the same as

$$(x \quad y)\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

44. Verify that the curl of the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ can be written

$$\text{curl } \mathbf{F} = \begin{pmatrix} 0 & -\partial/\partial x & \partial/\partial x \\ \partial/\partial x & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{pmatrix}\begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

(Readers who are not familiar with the concept of the curl of a vector field should see Section 9.7.)

45. As shown in Figure 8.1(a), a spacecraft can perform rotations called **pitch**, **roll**, and **yaw** about three distinct axes. To describe the coordinates of a point P we use two coordinate systems: a fixed three-dimensional Cartesian coordinate system in which the coordinates of P are (x, y, z) and a spacecraft coordinate system that moves with the particular rotation. In Figure 8.1(b) we have illustrated a yaw—that is, a rotation around the z-axis (which is perpendicular to the plane of the paper). The coordinates (x_Y, y_Y, z_Y) of the point P in the spacecraft system after the yaw are related to the coordinates (x, y, z) of P in the fixed coordinate system by the equations

$$\begin{aligned} x_Y &= x\cos\gamma + y\sin\gamma \\ y_Y &= -x\sin\gamma + y\cos\gamma \\ z_Y &= z \end{aligned}$$

where γ is the angle of rotation.

(a) Verify that the foregoing system of equations can be written as the matrix equation

$$\begin{pmatrix} x_Y \\ y_Y \\ z_Y \end{pmatrix} = \mathbf{M}_Y\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $\mathbf{M}_Y = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Gauss-Jordan elimination can require about 50% more operations than Gaussian elimination.

(*iii*) A system of linear equations that has more equations than unknowns is said to be **overdetermined**, whereas a system that has fewer equations than unknowns is called **underdetermined**. As a rule, an overdetermined system is usually—not always—inconsistent, and an underdetermined system is usually—not always—consistent. (See Examples 7 and 9.) It should be noted that it is impossible for a consistent underdetermined system to possess a single or unique solution. To see this, suppose we have m equations and n unknowns where $m < n$. If Gaussian elimination is used to solve such a system, then the row-echelon form that is row equivalent to the matrix of the system will contain $r \le m$ nonzero rows. Thus we can solve for r of the variables in terms of $n - r > 0$ variables. If the underdetermined system is consistent, then those remaining $n - r$ variables can be chosen arbitrarily, and so the system has an infinite number of solutions.

EXERCISES 8.2

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–20, use either Gaussian elimination or Gauss-Jordan elimination to solve the given system or show that no solution exists.

1. $\begin{aligned} x_1 - x_2 &= 11 \\ 4x_1 + 3x_2 &= -5 \end{aligned}$

2. $\begin{aligned} 3x_1 - 2x_2 &= 4 \\ x_1 - x_2 &= -2 \end{aligned}$

3. $\begin{aligned} 9x_1 + 3x_2 &= -5 \\ 2x_1 + x_2 &= -1 \end{aligned}$

4. $\begin{aligned} 10x_1 + 15x_2 &= 1 \\ 3x_1 + 2x_2 &= -1 \end{aligned}$

5. $\begin{aligned} x_1 - x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= -11 \end{aligned}$

6. $\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 9 \\ x_1 - x_2 + x_3 &= 3 \end{aligned}$

7. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$

8. $\begin{aligned} x_1 + 2x_2 - 4x_3 &= 9 \\ 5x_1 - x_2 + 2x_3 &= 1 \end{aligned}$

9. $\begin{aligned} x_1 - x_2 - x_3 &= 8 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$

10. $\begin{aligned} 3x_1 + x_2 &= 4 \\ 4x_1 + 3x_2 &= -3 \\ 2x_1 - x_2 &= 11 \end{aligned}$

11. $\begin{aligned} 2x_1 + 2x_2 &= 0 \\ -2x_1 + x_2 + x_3 &= 0 \\ 3x_1 + x_3 &= 0 \end{aligned}$

12. $\begin{aligned} x_1 - x_2 - 2x_3 &= 0 \\ 2x_1 + 4x_2 + 5x_3 &= 0 \\ 6x_1 - 3x_3 &= 0 \end{aligned}$

13. $\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 - 3x_2 - x_3 &= 0 \end{aligned}$

14. $\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 + x_3 &= 1 \end{aligned}$

15. $\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 - x_3 &= -1 \\ 3x_1 + x_2 + x_3 &= 5 \end{aligned}$

16. $\begin{aligned} x_1 - x_2 - 2x_3 &= -1 \\ -3x_1 - 2x_2 + x_3 &= -7 \\ 2x_1 + 3x_2 + x_3 &= 8 \end{aligned}$

17. $\begin{aligned} x_1 + x_3 - x_4 &= 1 \\ 2x_2 + x_3 + x_4 &= 3 \\ x_1 - x_2 + x_4 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 2 \end{aligned}$

18. $\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ 3x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 2x_2 + 2x_3 + 3x_4 &= 3 \\ 4x_1 + 5x_2 - 2x_3 + x_4 &= 16 \end{aligned}$

19. $\begin{aligned} x_2 + x_3 - x_4 &= 4 \\ x_1 + 3x_2 + 5x_3 - x_4 &= 1 \\ x_1 + 2x_2 + 5x_3 - 4x_4 &= -2 \\ x_1 + 4x_2 + 6x_3 - 2x_4 &= 6 \end{aligned}$

20. $\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ 4x_1 + 9x_2 + x_3 + 12x_4 &= 0 \\ 3x_1 + 9x_2 + 6x_3 + 21x_4 &= 0 \\ x_1 + 3x_2 + x_3 + 9x_4 &= 0 \end{aligned}$

In Problems 21 and 22, use a calculator to solve the given system.

21. $\begin{aligned} x_1 + x_2 + x_3 &= 4.280 \\ 0.2x_1 - 0.1x_2 - 0.5x_3 &= -1.978 \\ 4.1x_1 + 0.3x_2 + 0.12x_3 &= 1.686 \end{aligned}$

22. $\begin{aligned} 2.5x_1 + 1.4x_2 + 4.5x_3 &= 2.6170 \\ 1.35x_1 + 0.95x_2 + 1.2x_3 &= 0.7545 \\ 2.7x_1 + 3.05x_2 - 1.44x_3 &= -1.4292 \end{aligned}$

In Problems 23–28, use the procedures illustrated in Example 10 to balance the given chemical equation.

23. $Na + H_2O \rightarrow NaOH + H_2$

24. $KClO_3 \rightarrow KCl + O_2$

25. $Fe_3O_4 + C \rightarrow Fe + CO$

26. $C_5H_8 + O_2 \rightarrow CO_2 + H_2O$

27. $Cu + HNO_3 \rightarrow Cu(NO_3)_2 + H_2O + NO$

28. $Ca_3(PO_4)_2 + H_3PO_4 \rightarrow Ca(H_2PO_4)_2$

In Problems 29 and 30, set up and solve the system of equations for the currents in the branches of the given network.

29.

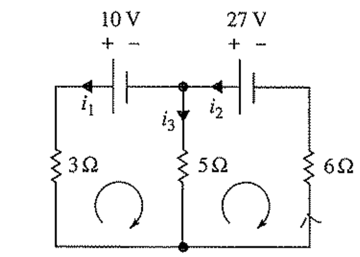

Figure 8.4 Network in Problem 29

30.

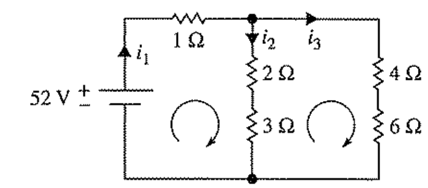

Figure 8.5 Network in Problem 30

An **elementary matrix E** is one obtained by performing a single row operation on the identity matrix **I**. In Problems 31–34, verify that the given matrix is an elementary matrix.

31. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

32. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$

33. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$

34. $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

If a matrix **A** is premultiplied by an elementary matrix **E**, the product **EA** will be that matrix obtained from **A** by performing the elementary row operation symbolized by **E**. In Problems 35–38, compute the given product for an arbitrary 3×3 matrix **A**.

35. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{A}$

36. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}\mathbf{A}$

37. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}\mathbf{A}$

38. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}\mathbf{A}$

The linear system (1) can be written as the matrix equation $\mathbf{AX} = \mathbf{B}$. Suppose $m = n$. If the $n \times n$ coefficient matrix **A** in the system has an LU-factorization $\mathbf{A} = \mathbf{LU}$ (see page 351), then the system $\mathbf{AX} = \mathbf{B}$, or $\mathbf{LUX} = \mathbf{B}$, can be solved efficiently in two steps *without* Gaussian or Gauss-Jordan elimination:

(*i*) First, let $\mathbf{Y} = \mathbf{UX}$ and solve $\mathbf{LY} = \mathbf{B}$ for **Y** by forward-substitution.

(*ii*) Then solve $\mathbf{UX} = \mathbf{Y}$ for **X** using back-substitution.

In Problems 39–42, use the results of Problem 46 in Exercises 8.1 to solve the given system.

39. $\begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

40. $\begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

41. $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 6 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

42. $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

Computer Lab Assignments

In Problems 43–46, use a CAS to solve the given system.

43. $\begin{aligned} 1.567x_1 - 3.48x_2 + 5.22x_3 &= 1.045 \\ 3.56x_1 + 4.118x_2 + 1.57x_3 &= -1.625 \end{aligned}$

44. $\begin{aligned} x_1 + 2x_2 - 2x_3 &= 0 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 - 6x_2 + 4x_3 &= 0 \\ 4x_1 + 14x_2 - 13x_3 &= 0 \end{aligned}$

45. $\begin{aligned} 1.2x_1 + 3.5x_2 - 4.4x_3 + 3.1x_4 &= 1.8 \\ 0.2x_1 - 6.1x_2 - 2.3x_3 + 5.4x_4 &= -0.6 \\ 3.3x_1 - 3.5x_2 - 2.4x_3 - 0.1x_4 &= 2.5 \\ 5.2x_1 + 8.5x_2 - 4.4x_3 - 2.9x_4 &= 0 \end{aligned}$

46. $\begin{aligned} x_1 - x_2 - x_3 + 2x_4 - x_5 &= 5 \\ 6x_1 + 9x_2 - 6x_3 + 17x_4 - x_5 &= 40 \\ 2x_1 + x_2 - 2x_3 + 5x_4 - x_5 &= 12 \\ x_1 + 2x_2 - x_3 + 3x_4 &= 7 \\ x_1 + 2x_2 + x_3 + 3x_4 &= 1 \end{aligned}$

EXERCISES 8.3

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–10, use (*iii*) of Theorem 8.4 to find the rank of the given matrix.

1. $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 4 \\ -1 & 0 & 3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 3 & -1 & 2 & 0 \\ 6 & 2 & 4 & 5 \end{pmatrix}$

7. $\begin{pmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{pmatrix}$

8. $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{pmatrix}$

9. $\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 1 & 0 & 5 & 1 \\ 2 & 1 & \frac{2}{3} & 3 & \frac{1}{3} \\ 6 & 6 & 6 & 12 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & -2 & 1 & 8 & -1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & -1 & 2 & 10 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 8 & -1 & 1 & 2 & 6 \end{pmatrix}$

In Problems 11–14, determine whether the given set of vectors is linearly dependent or linearly independent.

11. $\mathbf{u}_1 = \langle 1, 2, 3\rangle$, $\mathbf{u}_2 = \langle 1, 0, 1\rangle$, $\mathbf{u}_3 = \langle 1, -1, 5\rangle$

12. $\mathbf{u}_1 = \langle 2, 6, 3\rangle$, $\mathbf{u}_2 = \langle 1, -1, 4\rangle$, $\mathbf{u}_3 = \langle 3, 2, 1\rangle$, $\mathbf{u}_4 = \langle 2, 5, 4\rangle$

13. $\mathbf{u}_1 = \langle 1, -1, 3, -1\rangle$, $\mathbf{u}_2 = \langle 1, -1, 4, 2\rangle$, $\mathbf{u}_3 = \langle 1, -1, 5, 7\rangle$

14. $\mathbf{u}_1 = \langle 2, 1, 1, 5\rangle$, $\mathbf{u}_2 = \langle 2, 2, 1, 1\rangle$, $\mathbf{u}_3 = \langle 3, -1, 6, 1\rangle$, $\mathbf{u}_4 = \langle 1, 1, 1, -1\rangle$

15. Suppose the system $\mathbf{AX} = \mathbf{B}$ is consistent and $\mathbf{A}$ is a 5×8 matrix and rank($\mathbf{A}$) = 3. How many parameters does the solution of the system have?

16. Let $\mathbf{A}$ be a nonzero 4×6 matrix.
    (a) What is the maximum rank that $\mathbf{A}$ can have?
    (b) If rank($\mathbf{A}|\mathbf{B}$) = 2, then for what value(s) of rank($\mathbf{A}$) is the system $\mathbf{AX} = \mathbf{B}$, $\mathbf{B} \neq \mathbf{0}$, inconsistent? Consistent?
    (c) If rank($\mathbf{A}$) = 3, then how many parameters does the solution of the system $\mathbf{AX} = \mathbf{0}$ have?

17. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the first, second, and third column vectors, respectively, of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 7 \\ 1 & 0 & 2 \\ -1 & 5 & 13 \end{pmatrix}.$$

What can we conclude about rank($\mathbf{A}$) from the observation $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$?

[*Hint:* Read the *Remarks* at the end of this section.]

Discussion Problems

18. Suppose the system $\mathbf{AX} = \mathbf{B}$ is consistent and $\mathbf{A}$ is a 6×3 matrix. Suppose the maximum number of linearly independent rows in $\mathbf{A}$ is 3. Discuss: Is the solution of the system unique?

19. Suppose we wish to determine whether the set of column vectors

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$
$$\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 1 \\ 7 \\ -5 \\ 1 \end{pmatrix}$$

is linearly dependent or linearly independent. By Definition 7.7, if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0} \tag{4}$$

only for $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5 = 0$, then the set of vectors is linearly independent; otherwise the set is linearly dependent. But (4) is equivalent to the linear system

$$\begin{aligned} 4c_1 + c_2 - c_3 + 2c_4 + c_5 &= 0 \\ 3c_1 + 2c_2 + c_3 + 3c_4 + 7c_5 &= 0 \\ 2c_1 + 2c_2 + c_3 + 4c_4 - 5c_5 &= 0 \\ c_1 + c_2 + c_3 + c_4 + c_5 &= 0. \end{aligned}$$

Without doing any further work, explain why we can now conclude that the set of vectors is linearly dependent.

Computer Lab Assignments

20. A CAS can be used to row reduce a matrix to a row-echelon form. Use a CAS to determine the ranks of the augmented matrix ($\mathbf{A}|\mathbf{B}$) and the coefficient matrix $\mathbf{A}$ for

$$\begin{aligned} x_1 + 2x_2 - 6x_3 + x_4 + x_5 + x_6 &= 2 \\ 5x_1 + 2x_2 - 2x_3 + 5x_4 + 4x_5 + 2x_6 &= 3 \\ 6x_1 + 2x_2 - 2x_3 + x_4 + x_5 + 3x_6 &= -1 \\ -x_1 + 2x_2 + 3x_3 + x_4 - x_5 + 6x_6 &= 0 \\ 9x_1 + 7x_2 - 2x_3 + x_4 + 4x_5 &= 5. \end{aligned}$$

Is the system consistent or inconsistent? If consistent, solve the system.

8.4 Determinants

Introduction Suppose $\mathbf{A}$ is an $n \times n$ matrix. Associated with $\mathbf{A}$ is a *number* called the **determinant of A** and is denoted by det $\mathbf{A}$. Symbolically, we distinguish a matrix $\mathbf{A}$ from the determinant of $\mathbf{A}$ by replacing the parentheses by vertical bars:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{n2} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{n2} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

A determinant of an $n \times n$ matrix is said to be **a determinant of order *n***. We begin by defining the determinants of 1×1, 2×2, and 3×3 matrices.

A Definition For a 1×1 matrix $\mathbf{A} = (a)$, we have det $\mathbf{A} = |a| = a$. For example, if $\mathbf{A} = (-5)$, then det $\mathbf{A} = |-5| = -5$. In this case the vertical bars $|\,|$ around a number *do not* mean absolute value of the number.

DEFINITION 8.9 Determinant of a 2×2 Matrix

The determinant of $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is

$$\det\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{1}$$

As a mnemonic, a determinant of order 2 is thought to be the product of the main diagonal entries of $\mathbf{A}$ minus the product of the other diagonal entries:

multiply   multiply subtract

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{2}$$

For example, if $\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 5 & 9 \end{pmatrix}$, then det $\mathbf{A} = \begin{vmatrix} 6 & -3 \\ 5 & 9 \end{vmatrix} = 6(9) - (-3)(5) = 69$.

EXERCISES 8.4

Answers to selected odd numbered problems begin on page ANS-17.

In Problems 1–4, suppose

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \\ -2 & 3 & 5 \end{pmatrix}$$

Evaluate the indicated minor determinant or cofactor.

1. M_{12} 2. M_{32} 3. C_{13} 4. C_{22}

In Problems 5–8, suppose

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 4 & 0 \\ 1 & 2 & -2 & 3 \\ 5 & 1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Evaluate the indicated minor determinant or cofactor.

5. M_{33} 6. M_{41} 7. C_{34} 8. C_{23}

In Problems 9–14, evaluate the determinant of the given matrix.

9. (-7)

10. (2)

11. $\begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}$

12. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{pmatrix}$

13. $\begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix}$

14. $\begin{pmatrix} -3-\lambda & -4 \\ -2 & 5-\lambda \end{pmatrix}$

In Problems 15–28, evaluate the determinant of the given matrix by cofactor expansion.

15. $\begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{pmatrix}$

16. $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

17. $\begin{pmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{pmatrix}$

18. $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & 9 \end{pmatrix}$

19. $\begin{pmatrix} 4 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

20. $\begin{pmatrix} \frac{1}{4} & 6 & 0 \\ \frac{1}{3} & 8 & 0 \\ \frac{1}{2} & 9 & 0 \end{pmatrix}$

21. $\begin{pmatrix} -2 & -1 & 4 \\ -3 & 6 & 1 \\ -3 & 4 & 8 \end{pmatrix}$

22. $\begin{pmatrix} 3 & 5 & 1 \\ -1 & 2 & 5 \\ 7 & -4 & 10 \end{pmatrix}$

23. $\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 2 & 3 & 4 \end{pmatrix}$

24. $\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ 2+x & 3+y & 4+z \end{pmatrix}$

25. $\begin{pmatrix} 1 & 1 & -3 & 0 \\ 1 & 5 & 3 & 2 \\ 1 & -2 & 1 & 0 \\ 4 & 8 & 0 & 0 \end{pmatrix}$

26. $\begin{pmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{pmatrix}$

27. $\begin{pmatrix} 3 & 2 & 0 & 1 & -1 \\ 0 & 1 & 4 & 2 & 3 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

28. $\begin{pmatrix} 2 & 2 & 0 & 0 & -2 \\ 1 & 1 & 6 & 0 & 5 \\ 1 & 0 & 2 & -1 & -1 \\ 2 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$

In Problems 29 and 30, find the values of λ that satisfy the given equation.

29. $\begin{vmatrix} -3-\lambda & 10 \\ 2 & 5-\lambda \end{vmatrix} = 0$

30. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 3 & 3 & -\lambda \end{vmatrix} = 0$

8.5 Properties of Determinants

Introduction In this section we are going to consider some of the many properties of determinants. Our goal in the discussion is to use these properties to develop a means of evaluating a determinant that is an alternative to cofactor expansion.

Properties The first property states that the determinant of an $n \times n$ matrix and its transpose are the same.

THEOREM 8.8 Determinant of a Transpose

If $\mathbf{A}^T$ is the transpose of the $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{A}^T = \det \mathbf{A}$.

For example, for the matrix $\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 3 & -4 \end{pmatrix}$, we have $\mathbf{A}^T = \begin{pmatrix} 5 & 3 \\ 7 & -4 \end{pmatrix}$. Observe that

$$\det \mathbf{A} = \begin{vmatrix} 5 & 7 \\ 3 & -4 \end{vmatrix} = -41 \quad \text{and} \quad \det \mathbf{A}^T = \begin{vmatrix} 5 & 3 \\ 7 & -4 \end{vmatrix} = -41.$$

Since transposing a matrix interchanges its rows and columns, the significance of Theorem 8.8 is that statements concerning determinants and the rows of a matrix also hold when the word "row" is replaced by the word "column."

THEOREM 8.9 Two Identical Rows

If any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$ are the same then $\det \mathbf{A} = 0$.

Example 1 Matrix with Two Identical Rows

Since the second and third columns in the matrix $\mathbf{A} = \begin{pmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{pmatrix}$ are the same, it follows from Theorem 8.9 that

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{vmatrix} = 0.$$

You should verify this by expanding the determinant by cofactors. ❑

THEOREM 8.10 Zero Row or Column

If all the entries in a row (column) of an $n \times n$ matrix $\mathbf{A}$ are zero, then $\det \mathbf{A} = 0$.

Proof Suppose the ith row of $\mathbf{A}$ consists of all zeros. Hence all the products in the expansion of $\det \mathbf{A}$ by cofactors along the ith row are zero and consequently $\det \mathbf{A} = 0$. ❑

For example, it follows immediately from Theorem 8.10 that

$$\text{zero row} \rightarrow \begin{vmatrix} 0 & 0 \\ 7 & -6 \end{vmatrix} = 0 \quad \text{and} \quad \overset{\text{zero column} \downarrow}{\begin{vmatrix} 4 & 6 & 0 \\ 1 & 5 & 0 \\ 8 & -1 & 0 \end{vmatrix}} = 0.$$

THEOREM 8.11 Interchanging Rows

If $\mathbf{B}$ is the matrix obtained by interchanging any two rows (columns) of an $n \times n$ matrix $\mathbf{A}$, then $\det \mathbf{B} = -\det \mathbf{A}$.

(b) The determinant of the diagonal matrix $\mathbf{A} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{vmatrix} = (-3) \cdot 6 \cdot 4 = -72.$$ ❑

Row Reduction Evaluating the determinant of an $n \times n$ matrix by the method of cofactor expansion requires a herculean effort when the order of the matrix is large. To expand the determinant of, say, a 5×5 matrix with nonzero entries requires evaluating five cofactors that are determinants of 4×4 submatrices; each of these in turn requires four additional cofactors that are determinants of 3×3 submatrices, and so on. There is a more practical (and programmable) method for evaluating the determinant of a matrix. This method is based on **reducing** the matrix to a triangular form by row operations and the fact that determinants of triangular matrices are easy to evaluate (see Theorem 8.15).

Example 6 Reducing a Determinant to Triangular Form

Evaluate the determinant of $\mathbf{A} = \begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$.

Solution

$$\det \mathbf{A} = \begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 1 & 2 & 4 \end{vmatrix} \quad \text{(2 is a common factor in third row: Theorem 8.12)}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ -4 & -3 & 2 \\ 6 & 2 & 7 \end{vmatrix} \quad \text{(Interchange first and third rows: Theorem 8.11)}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 6 & 2 & 7 \end{vmatrix} \quad \text{(4 times first row added to second row: Theorem 8.14)}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & -10 & -17 \end{vmatrix} \quad \text{(}-6 \text{ times first row added to third row: Theorem 8.14)}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & 5 & 18 \\ 0 & 0 & 19 \end{vmatrix} \quad \text{(2 times second row added to third row: Theorem 8.14)}$$

$$= (-2)(1)(5)(19) = -190 \quad \text{(Theorem 8.15)}$$ ❑

Our final theorem concerns cofactors. We saw in Section 8.4 that a determinant det **A** of an $n \times n$ matrix **A** could be evaluated by cofactor expansion along any row (column). This means that the n entries a_{ij} of a row (column) are multiplied by the corresponding cofactors C_{ij} and the n products are added. If, however, the entries a_{ij} of a row (a_{ij} of a column) of **A** are multiplied by the corresponding cofactors C_{kj} of a different row (C_{ik} of a different column), the sum of the n products is zero.

THEOREM 8.16 A Property of Cofactors

Suppose **A** is an $n \times n$ matrix. If $a_{i1}, a_{i2}, \ldots, a_{in}$ are the entries in the ith row and $C_{k1}, C_{k2}, \ldots, C_{kn}$ are the cofactors of the entries in the kth row, then

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \quad \text{for } i \neq k.$$

If $a_{1j}, a_{2j}, \ldots, a_{nj}$ are the entries in the jth column and $C_{1k}, C_{2k}, \ldots, C_{nk}$ are the cofactors of the entries in the kth column, then

$$a_{1j}C_{1k} + a_{2j}C_{2k} + \cdots + a_{nj}C_{nk} = 0 \quad \text{for } j \neq k.$$

Proof We shall prove the result for rows. Let **B** be the matrix obtained from **A** by letting the entries in the ith row of **A** be the same as the entries in the kth row—that is, $a_{i1} = a_{k1}, a_{i2} = a_{k2}, \ldots, a_{in} = a_{kn}$. Since **B** has two rows that are the same, it follows from Theorem 8.9 that det **B** = 0. Cofactor expansion along the kth row then gives the desired result:

$$\begin{aligned} 0 = \det \mathbf{B} &= a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} \\ &= a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn}. \end{aligned}$$ ❑

Example 7 Cofactors of Third Row/Entries of First Row

Consider the matrix $\mathbf{A} = \begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$. Suppose we then multiply the entries of the first row by the cofactors of the third row and add the results; that is,

$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 6\begin{vmatrix} 2 & 7 \\ -3 & 2 \end{vmatrix} + 2\left(-\begin{vmatrix} 6 & 7 \\ -4 & 2 \end{vmatrix}\right) + 7\begin{vmatrix} 6 & 2 \\ -4 & -3 \end{vmatrix}$$

$$= 6(25) + 2(-40) + 7(-10) = 0.$$ ❑

EXERCISES 8.5

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–10, state the appropriate theorem(s) in this section that justifies the given equality. Do not expand the determinants by cofactors.

1. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$

3. $\begin{vmatrix} -5 & 6 \\ 2 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -6 & -8 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 18 \\ 5 & 9 & -12 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 5 & 9 & -4 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 18 \\ 5 & 9 & -12 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 18 \\ 5 & 9 & -12 \\ 1 & 2 & 3 \end{vmatrix}$

7. $\begin{vmatrix} 0 & 5 & 0 & 6 \\ 2 & 1 & 0 & 8 \\ 0 & 2 & 0 & -9 \\ 0 & 6 & 0 & 4 \end{vmatrix} = 0$

8. $\begin{vmatrix} 3 & 2 & 1 \\ 2 & 6 & 3 \\ 5 & -8 & -4 \end{vmatrix} = 0$

9. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$.

10. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix}$

In Problems 11–14, evaluate the determinant of the given matrix using the result

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5.$$

11. $\mathbf{A} = \begin{pmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{pmatrix}$

12. $\mathbf{B} = \begin{pmatrix} 2a_1 & a_2 & a_3 \\ 6b_1 & 3b_2 & 3b_3 \\ 2c_1 & c_2 & c_3 \end{pmatrix}$

13. $\mathbf{C} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ b_1 & b_2 & b_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{pmatrix}$

14. $\mathbf{D} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

In Problems 15–18, evaluate the determinant of the given matrix without expanding by cofactors.

15. $\mathbf{A} = \begin{pmatrix} 6 & 1 & 8 & 10 \\ 0 & \frac{2}{3} & 7 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -5 \end{pmatrix}$

16. $\mathbf{B} = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

17. $\mathbf{C} = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

18. $\mathbf{D} = \begin{pmatrix} 0 & 7 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

In Problems 19 and 20, verify that $\det \mathbf{A} = \det \mathbf{A}^T$ for the given matrix **A**.

19. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & -1 \\ 1 & 2 & -1 \end{pmatrix}$

20. $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 7 & 2 & -1 \end{pmatrix}$

21. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 8 \\ 0 & -1 & 0 \end{pmatrix}.$$

Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

22. Suppose **A** is an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{A}^2 = \mathbf{AA}$. Show that $\det \mathbf{A} = \pm 1$.

23. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{pmatrix}.$$

Without expanding, evaluate $\det \mathbf{A}$.

24. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{pmatrix}.$$

Without expanding, show that $\det \mathbf{A} = 0$.

In Problems 25–32, use the procedure illustrated in Example 6 to evaluate the determinant of the given matrix.

25. $\begin{pmatrix} 1 & 1 & 5 \\ 4 & 3 & 6 \\ 0 & -1 & 1 \end{pmatrix}$

26. $\begin{pmatrix} 2 & 4 & 5 \\ 4 & 2 & 0 \\ 8 & 7 & -2 \end{pmatrix}$

27. $\begin{pmatrix} -1 & 2 & 3 \\ 4 & -5 & -2 \\ 9 & -9 & 6 \end{pmatrix}$

28. $\begin{pmatrix} -2 & 2 & -6 \\ 5 & 0 & 1 \\ 1 & -2 & 2 \end{pmatrix}$

29. $\begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{pmatrix}$

30. $\begin{pmatrix} 0 & 1 & 4 & 5 \\ 2 & 5 & 0 & 1 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 3 & 2 \end{pmatrix}$

31. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 5 & 8 & 20 \end{pmatrix}$

32. $\begin{pmatrix} 2 & 9 & 1 & 8 \\ 1 & 3 & 7 & 4 \\ 0 & 1 & 6 & 5 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

33. By proceeding as in Example 6, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

34. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$. [*Hint:* See Problem 33.]

In Problems 35 and 36, verify Theorem 8.16 by evaluating $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ and $a_{13}C_{12} + a_{23}C_{22} + a_{33}C_{32}$ for the given matrix.

35. $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 4 & -2 & 1 \end{pmatrix}$

36. $\mathbf{A} = \begin{pmatrix} 3 & 0 & 5 \\ -2 & 3 & -1 \\ 2 & 2 & -3 \end{pmatrix}$

37. Let $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 4 \\ -1 & -5 \end{pmatrix}$. Verify that

$$\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}.$$

38. Suppose **A** is a 5×5 matrix for which $\det \mathbf{A} = -7$. What is the value of $\det(2\mathbf{A})$?

39. An $n \times n$ matrix **A** is said to be **skew-symmetric** if $\mathbf{A}^T = -\mathbf{A}$. If **A** is a 5×5 skew-symmetric matrix, show that $\det \mathbf{A} = 0$.

40. It takes about $n!$ multiplications to evaluate the determinant of an $n \times n$ matrix using expansion by cofactors, whereas it takes about $n^3/3$ arithmetic operations using the row reduction method illustrated in Example 6.

(a) Compare the number of operations for both methods using a 25×25 matrix.

(b) If a computer can do 50,000 operations per second, compare the lengths of time it would take the computer to evaluate the determinant of a 25×25 matrix using cofactor expansion and row reduction.

8.6 Inverse of a Matrix

Introduction The concept of the determinant of an $n \times n$, or square, matrix will play an important role in this and the following section.

8.6.1 Finding the Inverse

In the real number system, if a is a nonzero number then there exists a number b such that $ab = ba = 1$. The number b is called the multiplicative inverse of the number a and is denoted by a^{-1}. For a square matrix **A** it is also important to know whether we can find another square matrix **B** of the same order such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. We have the following definition.

DEFINITION 8.11 Inverse of a Matrix

Let **A** be an $n \times n$ matrix. If there exists an $n \times n$ matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}, \tag{1}$$

where **I** is the $n \times n$ identity, then the matrix **A** is said to be **nonsingular** or **invertible**. The matrix **B** is said to be the **inverse of A**.

For example, the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is nonsingular or invertible since the matrix $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is its inverse. To verify this, observe that

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

and

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Unlike the real number system, where *every* nonzero number a has a multiplicative inverse, not every nonzero $n \times n$ matrix **A** has an inverse.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ 0 & 0 \end{pmatrix}.$$

Solution We found the inverse of the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}$$

in Example 4. Thus, (7) gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ 62 \\ -36 \end{pmatrix}.$$

Consequently, $x_1 = 19$, $x_2 = 62$, and $x_3 = -36$. ❑

Uniqueness When $\det \mathbf{A} \neq 0$ the solution of the system $\mathbf{AX} = \mathbf{B}$ is unique. Suppose not—that is, suppose $\det \mathbf{A} \neq 0$ and $\mathbf{X}_1$ and $\mathbf{X}_2$ are two *different* solution vectors. Then $\mathbf{AX}_1 = \mathbf{B}$ and $\mathbf{AX}_2 = \mathbf{B}$ imply $\mathbf{AX}_1 = \mathbf{AX}_2$. Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and so $\mathbf{A}^{-1}(\mathbf{AX}_1) = \mathbf{A}^{-1}(\mathbf{AX}_2)$ and $(\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_1 = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_2$. This gives $\mathbf{IX}_1 = \mathbf{IX}_2$ or $\mathbf{X}_1 = \mathbf{X}_2$, which contradicts our assumption that $\mathbf{X}_1$ and $\mathbf{X}_2$ were different solution vectors.

Homogeneous Systems A homogeneous system of equations can be written $\mathbf{AX} = \mathbf{0}$. Recall that a homogeneous system always possesses the trivial solution $\mathbf{X} = \mathbf{0}$ and possibly an infinite number of solutions. In the next theorem we shall see that a homogeneous system of n equations in n unknowns possesses *only* the trivial solution when $\mathbf{A}$ is nonsingular.

THEOREM 8.21 Trivial Solution Only

A homogeneous system of n linear equations in n unknowns $\mathbf{AX} = \mathbf{0}$ has only the trivial solution if and only if $\mathbf{A}$ is nonsingular.

Proof We prove the sufficiency part of the theorem. Suppose $\mathbf{A}$ is nonsingular. Then by (7), we have the unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$. ❑

The next theorem will answer the question: When does a homogeneous system of n linear equations in n unknowns possess a nontrivial solution? Bear in mind that if a homogeneous system has one nontrivial solution, it must have an infinite number of solutions.

THEOREM 8.22 Existence of Nontrivial Solutions

A homogeneous system of n linear equations in n unknowns $\mathbf{AX} = \mathbf{0}$ has a nontrivial solution if and only if $\mathbf{A}$ is singular.

In view of Theorem 8.22, we can conclude that a homogeneous system of n linear equations in n unknowns $\mathbf{AX} = \mathbf{0}$ possesses

- only the trivial solution if and only if $\det \mathbf{A} \neq 0$, and
- a nontrivial solution if and only if $\det \mathbf{A} = 0$.

The last result will be put to use in Section 8.8.

Remarks

(*i*) As a practical means of solving n linear equations in n unknowns, the use of an inverse matrix offers few advantages over the method of Section 8.2. However, in applications we sometimes need to solve a system $\mathbf{AX} = \mathbf{B}$ several times; that is, we need to examine the solutions of the system corresponding to the same coefficient matrix $\mathbf{A}$ but different input vectors $\mathbf{B}$. In this case, the single calculation of $\mathbf{A}^{-1}$ enables us to obtain these solutions quickly through the matrix multiplication $\mathbf{A}^{-1}\mathbf{B}$.

(*ii*) In Definition 8.11 we saw that if $\mathbf{A}$ is an $n \times n$ matrix and there exists another $n \times n$ matrix $\mathbf{B}$ that commutes with $\mathbf{A}$ such that

$$\mathbf{AB} = \mathbf{I} \quad \text{and} \quad \mathbf{BA} = \mathbf{I}, \tag{8}$$

then $\mathbf{B}$ is the inverse of $\mathbf{A}$. Although matrix multiplication is in general not commutative, the condition in (8) can be relaxed somewhat in this sense: If we find an $n \times n$ matrix $\mathbf{B}$ for which $\mathbf{AB} = \mathbf{I}$, then it can be proved that $\mathbf{BA} = \mathbf{I}$ as well, and so $\mathbf{B}$ is the inverse of $\mathbf{A}$. As a consequence of this result, in the subsequent sections of this chapter if we wish to prove that a certain matrix $\mathbf{B}$ is the inverse of a given matrix $\mathbf{A}$, it will suffice to show only that $\mathbf{AB} = \mathbf{I}$. We need not demonstrate that $\mathbf{B}$ commutes with $\mathbf{A}$ to give $\mathbf{I}$.

EXERCISES 8.6

Answers to selected odd-numbered problems begin on page ANS-17.

8.6.1 Finding the Inverse

In Problems 1 and 2, verify that the matrix $\mathbf{B}$ is the inverse of the matrix $\mathbf{A}$.

1. $\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{pmatrix}$

In Problems 3–14, use Theorem 8.19 to determine whether the given matrix is singular or nonsingular. If it is nonsingular, use Theorem 8.18 to find the inverse.

3. $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$

4. $\begin{pmatrix} \frac{1}{3} & -1 \\ 4 & 3 \end{pmatrix}$

5. $\begin{pmatrix} 6 & 0 \\ -3 & 2 \end{pmatrix}$

6. $\begin{pmatrix} -2\pi & -\pi \\ -\pi & \pi \end{pmatrix}$

7. $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 4 \\ 1 & -1 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 3 & 0 \\ 0 & 11 & 14 \\ -1 & 4 & 7 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{pmatrix}$

10. $\begin{pmatrix} 2 & -1 & 5 \\ 3 & 0 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

11. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

12. $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{pmatrix}$

13. $\begin{pmatrix} 0 & -1 & 1 & 4 \\ 3 & 2 & -2 & 1 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

In Problems 15–26, use Theorem 8.20 to find the inverse of the given matrix or show that no inverse exists.

15. $\begin{pmatrix} 6 & -2 \\ 0 & 4 \end{pmatrix}$

16. $\begin{pmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

17. $\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$

18. $\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix}$

19. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

20. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}$

21. $\begin{pmatrix} 4 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & -2 & 0 \end{pmatrix}$

22. $\begin{pmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ 8 & 10 & -6 \end{pmatrix}$

23. $\begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

24. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{pmatrix}$

25. $\begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

26. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

In Problems 27 and 28, use the given matrices to find $(\mathbf{AB})^{-1}$.

27. $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$, $\mathbf{B}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{5}{2} \end{pmatrix}$

28. $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 3 & -15 \\ 0 & -1 & 5 \\ -1 & -2 & 11 \end{pmatrix}$,

$\mathbf{B}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

29. If $\mathbf{A}^{-1} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, what is $\mathbf{A}$?

30. If $\mathbf{A}$ is nonsingular, then $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. Verify this for $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$.

31. Find a value of x such that the matrix $\mathbf{A} = \begin{pmatrix} 4 & -3 \\ x & -4 \end{pmatrix}$ is its own inverse.

32. Find the inverse of $\mathbf{A} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$.

33. A nonsingular matrix $\mathbf{A}$ is said to be **orthogonal** if $\mathbf{A}^{-1} = \mathbf{A}^T$.

(a) Verify that the matrix in Problem 32 is orthogonal.

(b) Verify that $\mathbf{A} = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$ is an orthogonal matrix.

34. Show that if $\mathbf{A}$ is an orthogonal matrix (see Problem 33), then $\det \mathbf{A} = \pm 1$.

35. If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices, use Theorem 8.19 to show that $\mathbf{AB}$ is nonsingular.

36. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. Show that if either $\mathbf{A}$ or $\mathbf{B}$ is singular, then $\mathbf{AB}$ is singular.

37. Show that if $\mathbf{A}$ is a nonsingular matrix, then $\det \mathbf{A}^{-1} = 1/\det \mathbf{A}$.

38. Show that if $\mathbf{A}^2 = \mathbf{A}$, then either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A}$ is singular.

39. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and $\mathbf{A}$ is nonsingular. Show that if $\mathbf{AB} = \mathbf{0}$, then $\mathbf{B} = \mathbf{0}$.

40. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and $\mathbf{A}$ is nonsingular. Show that if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

41. If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices, is $\mathbf{A} + \mathbf{B}$ necessarily nonsingular?

42. Consider the 3×3 diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Determine conditions such that $\mathbf{A}$ is nonsingular. If $\mathbf{A}$ is nonsingular, find $\mathbf{A}^{-1}$. Generalize your results to an $n \times n$ diagonal matrix.

8.6.2 Using the Inverse to Solve Systems

In Problems 43–50, use an inverse matrix to solve the given system of equations.

43. $\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 - x_2 &= 14 \end{aligned}$

44. $\begin{aligned} x_1 - x_2 &= 2 \\ 2x_1 + 4x_2 &= -5 \end{aligned}$

45. $\begin{aligned} 4x_1 - 6x_2 &= 6 \\ 2x_1 + x_2 &= 1 \end{aligned}$

46. $\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 4x_2 &= -3 \end{aligned}$

47. $\begin{aligned} x_1 + x_3 &= -4 \\ x_1 + x_2 + x_3 &= 0 \\ 5x_1 - x_2 &= 6 \end{aligned}$

48. $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -3 \end{aligned}$

49. $\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 - 2x_2 + 2x_3 &= -3 \\ 3x_1 - x_2 + 5x_3 &= 7 \end{aligned}$

50. $\begin{aligned} x_1 - x_3 &= 2 \\ x_2 + x_3 &= 1 \\ -x_1 + x_2 + 2x_3 + x_4 &= -5 \\ x_3 - x_4 &= 3 \end{aligned}$

In Problems 51 and 52, write the system in the form $\mathbf{AX} = \mathbf{B}$. Use $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ to solve the system for each matrix $\mathbf{B}$.

51. $\begin{aligned} 7x_1 - 2x_2 &= b_1, \\ 3x_1 - 2x_2 &= b_2, \end{aligned}$

$\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 10 \\ 50 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ -20 \end{pmatrix}$

52. $\begin{aligned} x_1 + 2x_2 + 5x_3 &= b_1 \\ 2x_1 + 3x_2 + 8x_3 &= b_2, \\ -x_1 + x_2 + 2x_3 &= b_3 \end{aligned}$

$\mathbf{B} = \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}$

In Problems 53–56, without solving, determine whether the given homogeneous system of equations has only the trivial solution or a nontrivial solution.

53. $\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 0 \\ 5x_1 + x_2 - 2x_3 &= 0 \end{aligned}$

54. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ -2x_1 + x_2 - 2x_3 &= 0 \end{aligned}$

55. $\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 5x_2 + 2x_4 &= 0 \\ x_1 + x_3 - x_4 &= 0 \\ 3x_1 + 2x_2 - x_3 + x_4 &= 0 \end{aligned}$

56. $\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - x_4 &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \\ x_2 - x_3 - x_4 &= 0 \end{aligned}$

57. The system of equations for the currents i_1, i_2, and i_3 in the network shown in Figure 8.6 is

$$\begin{aligned} i_1 + i_2 + i_3 &= 0 \\ -R_1 i_1 + R_2 i_2 \quad &= E_2 - E_1 \\ -R_2 i_2 + R_3 i_3 &= E_3 - E_2 \end{aligned}$$

where R_k and E_k, $k = 1, 2, 3$, are constants.

(a) Express the system as a matrix equation $\mathbf{AX} = \mathbf{B}$.

(b) Show that the coefficient matrix $\mathbf{A}$ is nonsingular.

(c) Use $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ to solve for the currents.

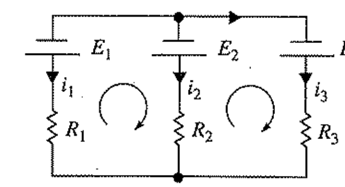

Figure 8.6 Network in Problem 57

58. Consider the square plate shown in Figure 8.7, with the temperatures as indicated on each side. Under some circumstances it can be shown that the approximate temperatures u_1, u_2, u_3, and u_4 at the points P_1, P_2, P_3, and P_4, respectively, are given by

$$u_1 = \frac{u_2 + u_4 + 100 + 100}{4}$$

$$u_2 = \frac{200 + u_3 + u_1 + 100}{4}$$

$$u_3 = \frac{200 + 100 + u_4 + u_2}{4}$$

$$u_4 = \frac{u_3 + 100 + 100 + u_1}{4}.$$

(a) Show that the above system can be written as the matrix equation

$$\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -200 \\ -300 \\ -300 \\ -200 \end{pmatrix}.$$

(b) Solve the system in part (a) by finding the inverse of the coefficient matrix.

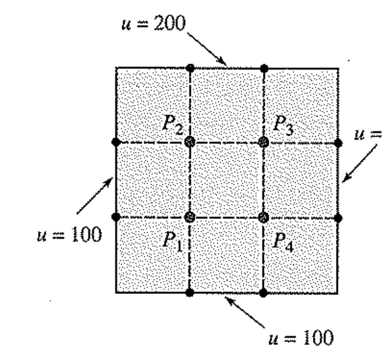

Figure 8.7 Plate in Problem 58

8.7 Cramer's Rule

Introduction We saw at the end of the preceding section that a system of n linear equations in n unknowns $\mathbf{AX} = \mathbf{B}$ has precisely one solution when $\det \mathbf{A} \neq 0$. This solution, as we shall now see, can be expressed in terms of determinants. For example, the system of two equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{1}$$

possesses the solution

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{2}$$

provided that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. The numerators and denominators in (2) are recognized as determinants. That is, the system (1) has the unique solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{3}$$