Hence $k_1 = k_3$ and $k_2 = k_3$. If $k_3 = 1$, then

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now for $\lambda_2 = 8$ we have

$$(\mathbf{A} - 8\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

In the equation $k_1 + k_2 + k_3 = 0$ we are free to select two of the variables arbitrarily. Choosing, on the one hand, $k_2 = 1$, $k_3 = 0$, and on the other, $k_2 = 0$, $k_3 = 1$, we obtain two linearly independent eigenvectors:

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

corresponding to a single eigenvalue. ❑

Complex Eigenvalues A matrix **A** may have **complex eigenvalues.**

THEOREM 8.24 Complex Eigenvalues and Eigenvectors

Let **A** be a square matrix with real entries. If $\lambda = \alpha + i\beta$, $\beta \neq 0$, is a complex eigenvalue of **A**, then its conjugate $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of **A**. If **K** is an eigenvector corresponding to λ, then its conjugate $\bar{\mathbf{K}}$ is an eigenvector corresponding to $\bar{\lambda}$.

Proof Since **A** is a matrix with real entries, the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is a polynomial equation with real coefficients. From algebra we know that complex roots of such equations appear in conjugate pairs. In other words, if $\lambda = \alpha + i\beta$ is a root, then $\bar{\lambda} = \alpha - i\beta$ is also a root. Now let **K** be an eigenvector of **A** corresponding to λ. By definition, $\mathbf{AK} = \lambda\mathbf{K}$. Taking complex conjugates of the latter equation gives

$$\overline{\mathbf{A}}\,\overline{\mathbf{K}} = \bar{\lambda}\overline{\mathbf{K}} \quad \text{or} \quad \mathbf{A}\overline{\mathbf{K}} = \bar{\lambda}\overline{\mathbf{K}},$$

since **A** is a real matrix. The last equation indicates that $\overline{\mathbf{K}}$ is an eigenvector corresponding to $\bar{\lambda}$. ❑

Example 5 Complex Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$.

Solution The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

From the quadratic formula, we find $\lambda_1 = 5 + 2i$ and $\lambda_2 = \bar{\lambda}_1 = 5 - 2i$.

Now for $\lambda_1 = 5 + 2i$, we must solve

$$\begin{aligned} (1 - 2i)k_1 - \quad k_2 &= 0 \\ 5k_1 - (1 + 2i)k_2 &= 0. \end{aligned}$$

Since $k_2 = (1 - 2i)k_1$,* it follows, after choosing $k_1 = 1$, that one eigenvector is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}.$$

From Theorem 8.24, we see that an eigenvector corresponding to $\lambda_2 = \bar{\lambda}_1 = 5 - 2i$ is

$$\mathbf{K}_2 = \overline{\mathbf{K}}_1 = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}.$$ ❑

Our last theorem follows immediately from the fact that the determinant of an upper triangular, lower triangular, or diagonal matrix is the product of the diagonal entries.

THEOREM 8.25 Triangular and Diagonal Matrices

The eigenvalues of an upper triangular, lower triangular, or diagonal matrix are the main diagonal entries.

*Note that the second equation is simply $1 + 2i$ times the first.

EXERCISES 8.8

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–6, determine which of the indicated column vectors are eigenvectors of the given matrix **A**. Give the corresponding eigenvalue.

1. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 - \sqrt{2} \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 2 + \sqrt{2} \\ 2 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$

3. $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$

4. $\mathbf{A} = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 2 + 2i \\ -1 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} 2 + 2i \\ 1 \end{pmatrix}$

5. $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

6. $\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$; $\mathbf{K}_1 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{K}_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

In Problems 7–22, find the eigenvalues and eigenvectors of the given matrix.

7. $\begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

9. $\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{pmatrix}$

11. $\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$

12. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

13. $\begin{pmatrix} 4 & 8 \\ 0 & -5 \end{pmatrix}$

14. $\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$

15. $\begin{pmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{pmatrix}$

16. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 0 & 4 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

19. $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

20. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

21. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -7 \end{pmatrix}$ 22. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of a nonsingular matrix $\mathbf{A}$. Furthermore, the eigenvectors for $\mathbf{A}$ and $\mathbf{A}^{-1}$ are the same. In Problems 23 and 24, verify these facts for the given matrix.

23. $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ 24. $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$

A matrix $\mathbf{A}$ is singular if and only if $\lambda = 0$ is an eigenvalue. In Problems 25 and 26, verify that the given matrix $\mathbf{A}$ is singular. Find the characteristic equation for $\mathbf{A}$ and verify that $\lambda = 0$ is an eigenvalue.

25. $\mathbf{A} = \begin{pmatrix} 6 & 0 \\ 3 & 0 \end{pmatrix}$ 26. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -4 & 5 \\ 7 & -4 & 8 \end{pmatrix}$

Computer Lab Assignments

27. A square matrix $\mathbf{A}$ is said to be a **stochastic matrix** if all its entries are nonnegative and the sum of the entries in each row (or the sum of the entries in each column) add up to 1. Stochastic matrices are important in probability theory.

(a) Verify that

$$\mathbf{A} = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}, \quad 0 \le p \le 1, 0 \le q \le 1,$$

and

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

are stochastic matrices.

(b) Use linear algebra software or a CAS to find the eigenvalues and eigenvectors of the 3×3 matrix $\mathbf{A}$ in part (a). Make up at least six more stochastic matrices of various sizes, 2×2, 3×3, 4×4, and 5×5. Find the eigenvalues and eigenvectors of each matrix. If you discern a pattern, form a conjecture and then try to prove it.

(c) For the 3×3 matrix $\mathbf{A}$ in part (a), use the software to find $\mathbf{A}^2, \mathbf{A}^3, \mathbf{A}^4, \ldots$ Repeat for the matrices that you constructed in part (b). If you discern a pattern, form a conjecture and then try to prove it.

8.9 Powers of Matrices

Introduction It is sometimes important to be able to quickly compute a power $\mathbf{A}^m$, m a positive integer, of an $n \times n$ matrix $\mathbf{A}$:

$$\mathbf{A}^m = \underbrace{\mathbf{AAA} \cdots \mathbf{A}}_{m \text{ factors}}.$$

Of course, computation of $\mathbf{A}^m$ could be done with the appropriate software or by writing a short computer program, but even then, you should be aware that it is inefficient to simply use brute force to carry out repeated multiplications: $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AA}^2$, $\mathbf{A}^4 = \mathbf{AAAA} = \mathbf{A}(\mathbf{A}^3) = \mathbf{A}^2\mathbf{A}^2$, and so on.

Computation of $\mathbf{A}^m$ We are going to sketch an alternative method for computing $\mathbf{A}^m$ by means of the following theorem known as the Cayley-Hamilton theorem.

THEOREM 8.26 Cayley-Hamilton Theorem

An $n \times n$ matrix $\mathbf{A}$ satisfies its own characteristic equation.

If $(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$ is the characteristic equation of $\mathbf{A}$, then Theorem 8.26 states that

$$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}. \tag{1}$$

Matrices of Order 2 The characteristic equation of the 2×2 matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ is $\lambda^2 - \lambda - 2 = 0$, and the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$. Theorem 8.26 implies $\mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = \mathbf{0}$, or, solving for the highest power of $\mathbf{A}$,

$$\mathbf{A}^2 = 2\mathbf{I} + \mathbf{A}. \tag{2}$$

Now if we multiply (2) by $\mathbf{A}$, we get $\mathbf{A}^3 = 2\mathbf{A} + \mathbf{A}^2$, and if we use (2) again to eliminate $\mathbf{A}^2$ on the right side of this new equation, then

$$\mathbf{A}^3 = 2\mathbf{A} + \mathbf{A}^2 = 2\mathbf{A} + (2\mathbf{I} + \mathbf{A}) = 2\mathbf{I} + 3\mathbf{A}.$$

Continuing in this manner—in other words, multiplying the last result by $\mathbf{A}$ and using (2) to eliminate $\mathbf{A}^2$—we obtain in succession powers of $\mathbf{A}$ expressed solely in terms of the identity matrix $\mathbf{I}$ and $\mathbf{A}$:

$$\begin{aligned} \mathbf{A}^4 &= 6\mathbf{I} + 5\mathbf{A} \\ \mathbf{A}^5 &= 10\mathbf{I} + 11\mathbf{A} \\ \mathbf{A}^6 &= 22\mathbf{I} + 21\mathbf{A} \end{aligned} \tag{3}$$

and so on (verify). Thus, for example,

$$\mathbf{A}^6 = 22\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 21\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -20 & 84 \\ -21 & 85 \end{pmatrix}. \tag{4}$$

Now we can determine the c_k without actually carrying out the repeated multiplications and resubstitutions as we did in (3). First, note that since the characteristic equation of the matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ can be written $\lambda^2 = 2 + \lambda$, results analogous to (3) must also hold for the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, that is, $\lambda^3 = 2 + 3\lambda$, $\lambda^4 = 6 + 5\lambda$, $\lambda^5 = 10 + 11\lambda$, $\lambda^6 = 22 + 21\lambda, \ldots$. It follows then that the equations

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} \quad \text{and} \quad \lambda^m = c_0 + c_1\lambda \tag{5}$$

hold *for the same pair of constants* c_0 and c_1. We can determine the constants c_0 and c_1 by simply setting $\lambda = -1$ and $\lambda = 2$ in the last equation in (5) and solving the resulting system of two equations in two unknowns. The solution of the system

$$\begin{aligned} (-1)^m &= c_0 + c_1(-1) \\ 2^m &= c_0 + c_1(2) \end{aligned}$$

is $c_0 = \frac{1}{3}[2^m + 2(-1)^m]$, $c_1 = \frac{1}{3}[2^m - (-1)^m]$. Now by substituting these coefficients in the first equation in (5), adding the two matrices and simplifying each entry, we obtain

$$\mathbf{A}^m = \begin{pmatrix} \frac{1}{3}[-2^m + 4(-1)^m] & \frac{4}{3}[2^m - (-1)^m] \\ -\frac{1}{3}[2^m - (-1)^m] & \frac{1}{3}[2^{m+2} - (-1)^m] \end{pmatrix}. \tag{6}$$

You should verify the result in (4) by setting $m = 6$ in (6). Note that (5) and (6) are valid for $m \ge 0$ since $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{A}^1 = \mathbf{A}$.

Matrices of Order n If the matrix $\mathbf{A}$ were 3×3, then the characteristic equation (1) is a cubic polynomial equation, and the analogue of (2) would enable us to express $\mathbf{A}^3$ in terms of $\mathbf{I}$, $\mathbf{A}$, and $\mathbf{A}^2$. We could proceed as just illustrated to write any power $\mathbf{A}^m$ in terms of $\mathbf{I}$, $\mathbf{A}$, and $\mathbf{A}^2$. In general, for an $n \times n$ matrix $\mathbf{A}$, we can write

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 + \cdots + c_{n-1}\mathbf{A}^{n-1},$$

and where each of the coefficients c_k, $k = 0, 1, \ldots, n-1$, depends on the value of m.

EXERCISES 8.12

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–20, determine whether the given matrix $\mathbf{A}$ is diagonalizable. If so, find the matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

1. $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

2. $\begin{pmatrix} -4 & -5 \\ 8 & 10 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} -9 & 13 \\ -2 & 6 \end{pmatrix}$

6. $\begin{pmatrix} -5 & -3 \\ 5 & 11 \end{pmatrix}$

7. $\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}$

8. $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$

9. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & -2 \\ -5 & 3 & 8 \end{pmatrix}$

13. $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 0 & -9 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

16. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

17. $\begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

18. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$

19. $\begin{pmatrix} -8 & -10 & 7 & -9 \\ 0 & 2 & 0 & 0 \\ -9 & -9 & 8 & -9 \\ 1 & 1 & -1 & 2 \end{pmatrix}$

20. $\begin{pmatrix} 4 & 2 & -1 & 4 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

In Problems 21–30, the given matrix $\mathbf{A}$ is symmetric. Find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ and the diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^T\mathbf{AP}$.

21. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$

23. $\begin{pmatrix} 5 & \sqrt{10} \\ \sqrt{10} & 8 \end{pmatrix}$

24. $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

25. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

26. $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

27. $\begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}$

28. $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

29. $\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$

30. $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

In Problems 31–34, use the procedure illustrated in Example 6 to identify the given conic section. Graph.

31. $5x^2 - 2xy + 5y^2 = 24$

32. $13x^2 - 10xy + 13y^2 = 288$

33. $-3x^2 + 8xy + 3y^2 = 20$

34. $16x^2 + 24xy + 9y^2 - 3x + 4y = 0$

35. Find a 2×2 matrix $\mathbf{A}$ that has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

36. Find a 3×3 symmetric matrix that has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$ and corresponding eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

37. If $\mathbf{A}$ is an $n \times n$ diagonalizable matrix, then $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{D}$ is a diagonal matrix. Show that if m is a positive integer, then $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$.

38. The mth power of a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

is

$$\mathbf{D}^m = \begin{pmatrix} a_{11}^m & 0 & \cdots & 0 \\ 0 & a_{22}^m & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{nn}^m \end{pmatrix}.$$

Use this result to compute

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}^4.$$

In Problems 39 and 40, use the results of Problems 37 and 38 to find the indicated power of the given matrix.

39. $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, $\mathbf{A}^5$

40. $\mathbf{A} = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}$, $\mathbf{A}^{10}$

8.13 Cryptography

Introduction The word *cryptography* is a combination of two Greek words: *crypto*, meaning "hidden" or "secret," and *grapho*, which means "writing." Cryptography then is the study of making "secret writings" or **codes**.

In this section we will consider a system of encoding and decoding messages that requires both the sender of the message and the receiver of the message to know:

- a specified rule of correspondence between a set of symbols (such as letters of the alphabet and punctuation marks from which messages are composed) and a set of integers; and
- a specified *nonsingular* matrix $\mathbf{A}$.

Encoding/Decoding A natural correspondence between the first twenty-seven nonnegative integers and the letters of the alphabet and a blank space (to separate words) is given by

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
space	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

(1)

From (1) the numerical equivalent of the message

SEND THE DOCUMENT TODAY

is

$$19\ 5\ 14\ 4\ 0\ 20\ 8\ 5\ 0\ 4\ 15\ 3\ 21\ 13\ 5\ 14\ 20\ 0\ 20\ 15\ 4\ 1\ 25. \qquad (2)$$

The sender will **encode** the message by means of the nonsingular matrix $\mathbf{A}$ and, as we shall see, the receiver of the encoded message will **decode** the message by means of the (unique) matrix $\mathbf{A}^{-1}$. The numerical message (2) is now written as a matrix. Since there are 23 symbols in the message, we need a matrix that will hold a minimum of 24 entries (an $m \times n$ matrix has mn entries). We choose to write (2) as the 3×8 matrix

$$\mathbf{M} = \begin{pmatrix} 19 & 5 & 14 & 4 & 0 & 20 & 8 & 5 \\ 0 & 4 & 15 & 3 & 21 & 13 & 5 & 14 \\ 20 & 0 & 20 & 15 & 4 & 1 & 25 & 0 \end{pmatrix}. \qquad (3)$$

Note that the last entry (a_{38}) in the message matrix $\mathbf{M}$ has been simply padded with a space represented by the number 0. Of course, we could have written (2) as a 6×4 or a 4×6 matrix but that would require a larger encoding matrix. A 3×8 matrix allows us to encode the message by means of a 3×3 matrix. The size of the matrices used is only a concern when the encoding and decoding are done by hand rather than by a computer.

The encoding matrix $\mathbf{A}$ is chosen, or rather constructed, so that

- $\mathbf{A}$ is nonsingular,
- $\mathbf{A}$ has only integer entries, and
- $\mathbf{A}^{-1}$ has only integer entries.

The last criterion is not particularly difficult to accomplish. We need only select the integer entries of $\mathbf{A}$ in such a manner that $\det \mathbf{A} = \pm 1$. For a 2×2 or a 3×3 matrix we can then find $\mathbf{A}^{-1}$ by the formulas in (4) and (5) of Section 8.6. If $\mathbf{A}$ has integer entries, then all the cofactors C_{11}, C_{12}, and so on are also integers. For the discussion on hand we choose

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix}. \qquad (4)$$

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student), the following statement is made:

It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \qquad (15)$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (15) could be considered reasonable. After simplification and some relabeling of the output, *Mathematica* yields the (approximate) general solution

$$y = c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x)$$
$$+ c_3e^{-0.476478x}\cos(0.759081x) + c_4e^{-0.476478x}\sin(0.759081x).$$

We note in passing that the **DSolve** and **dsolve** commands in *Mathematica* and *Maple*, like most aspects of any CAS, have their limitations.

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order differential equation, then to fit the general solution of the DE to the four initial conditions we must solve a system of four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 35, 36, 61, and 62 in Exercises 3.3.

*McGraw-Hill, New York, 1960.

Remarks

In case you are wondering, the method of this section also works for homogeneous linear *first-order* differential equations $ay' + by = 0$ with constant coefficients. For example, to solve, say, $2y' + 7y = 0$, we substitute $y = e^{mx}$ into the DE to obtain the auxiliary equation $2m + 7 = 0$. Using $m = -\frac{7}{2}$, the general solution of the DE is then $y = c_1e^{-7x/2}$.

EXERCISES 3.3

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–14, find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$
2. $y'' - 36y = 0$
3. $y'' - y' - 6y = 0$
4. $y'' - 3y' + 2y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 10y' + 25y = 0$
7. $12y'' - 5y' - 2y = 0$
8. $y'' + 4y' - y = 0$
9. $y'' + 9y = 0$
10. $3y'' + y = 0$
11. $y'' - 4y' + 5y = 0$
12. $2y'' + 2y' + y = 0$
13. $3y'' + 2y' + y = 0$
14. $2y'' - 3y' + 4y = 0$

In Problems 15–28, find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$
16. $y''' - y = 0$
17. $y''' - 5y'' + 3y' + 9y = 0$
18. $y''' + 3y'' - 4y' - 12y = 0$
19. $\frac{d^3u}{dt^3} + \frac{d^2u}{dt^2} - 2u = 0$
20. $\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$
21. $y''' + 3y'' + 3y' + y = 0$
22. $y''' - 6y'' + 12y' - 8y = 0$
23. $y^{(4)} + y''' + y'' = 0$
24. $y^{(4)} - 2y'' + y = 0$
25. $16\frac{d^4y}{dx^4} + 24\frac{d^2y}{dx^2} + 9y = 0$
26. $\frac{d^4y}{dx^4} - 7\frac{d^2y}{dx^2} - 18y = 0$
27. $\frac{d^5u}{dr^5} + 5\frac{d^4u}{dr^4} - 2\frac{d^3u}{dr^3} - 10\frac{d^2u}{dr^2} + \frac{du}{dr} + 5u = 0$
28. $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2} = 0$

In Problems 29–36, solve the given initial-value problem.

29. $y'' + 16y = 0, \quad y(0) = 2, y'(0) = -2$
30. $\frac{d^2y}{d\theta^2} + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, y'\left(\frac{\pi}{3}\right) = 2$
31. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$
32. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$
33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$
34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$
35. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$
36. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, y''(0) = 1$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$
38. $y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0$
39. $y'' + y = 0, \quad y'(0) = 0, y'\left(\frac{\pi}{2}\right) = 0$
40. $y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1$

In Problems 41 and 42, solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0, \quad y(0) = 1, y'(0) = 5$
42. $y'' - y = 0, \quad y(0) = 1, y'(1) = 0.$

In Problems 43–48, each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$ (b) $y'' + 4y = 0$
(c) $y'' + 2y' + y = 0$ (d) $y'' + y = 0$
(e) $y'' + 2y' + 2y = 0$ (f) $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

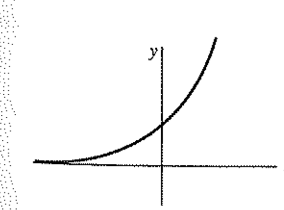

Figure 3.5 Graph for Problem 43

44.

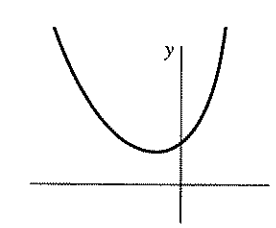

Figure 3.6 Graph for Problem 44

45.

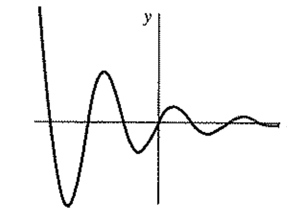

Figure 3.7 Graph for Problem 45

46.

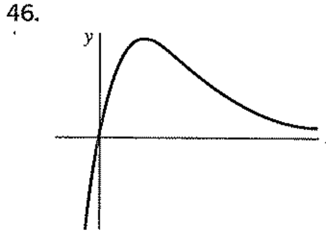

Figure 3.8 Graph for Problem 46

47.

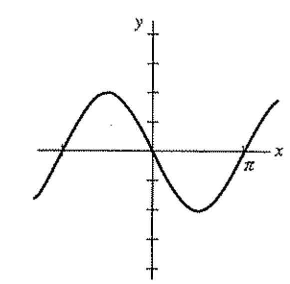

Figure 3.9 Graph for Problem 47

48.

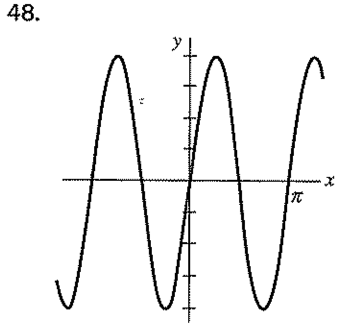

Figure 3.10 Graph for Problem 48

Discussion Problems

49. The roots of a cubic auxiliary equation are $m_1 = 4$ and $m_2 = m_3 = -5$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?
50. Two roots of a cubic auxiliary equation with real coefficients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation?
51. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x}\cos x$ is one solution.
52. To solve $y^{(4)} + y = 0$ we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS, but it can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.
53. Verify that $y = \sinh x - 2\cos\left(x + \frac{\pi}{6}\right)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.
54. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0, y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses (a) trivial solutions? (b) nontrivial solutions?
55. In the study of techniques of integration in calculus, certain indefinite integrals of the form $\int e^{ax} f(x)\,dx$ could be evaluated by applying integration by parts twice, recovering the original integral on the right-hand side, solving for the original integral, and obtaining a constant multiple $k\int e^{ax} f(x)\,dx$ on the left-hand side. Then the value of the integral is found by dividing by k.

EXERCISES 3.4

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–26, solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3 \sin 2x$
14. $y'' - 4y = (x^2 - 3) \sin 2x$
15. $y'' + y = 2x \sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x \cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3 \sin x)$
19. $y'' + 2y' + y = \sin x + 3 \cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
21. $y''' - 6y'' = 3 - \cos x$
22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
23. $y''' - 3y'' + 3y' - y = x - 4e^x$
24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
25. $y^{(4)} + 2y'' + y = (x - 1)^2$
26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36, solve the given initial-value problem.

27. $y'' + 4y = -2,\ y\left(\frac{\pi}{8}\right) = \frac{1}{2},\ y'\left(\frac{\pi}{8}\right) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11,$ $y(0) = 0,\ y'(0) = 0$
29. $5y'' + y' = -6x,\ y(0) = 0,\ y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x},\ y(0) = 2,\ y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x},\ y(0) = -3,\ y'(0) = 1$
32. $y'' - y = \cosh x,\ \ y(0) = 2,\ y'(0) = 12$
33. $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t,\ x(0) = 0,\ x'(0) = 0$
34. $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t,\ x(0) = 0,\ x'(0) = 0$
35. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x},\ y(0) = \frac{1}{2},\ y'(0) = \frac{5}{2},$ $y''(0) = -\frac{9}{2}$
36. $y''' + 8y = 2x - 5 + 8e^{-2x},\ y(0) = -5,\ y'(0) = 3,$ $y''(0) = -4$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' + y = x^2 + 1,\ y(0) = 5,\ y(1) = 0$
38. $y'' - 2y' + 2y = 2x - 2,\ y(0) = 0,\ y(\pi) = \pi$
39. $y'' + 3y = 6x,\ y(0) = 0,\ y(1) + y'(1) = 0$
40. $y'' + 3y = 6x,\ y(0) + y'(0) = 0,\ y(1) = 0$

In Problems 41 and 42, solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint:* Solve each problem on two intervals, and then find a solution so that y and y' are continuous at $x = \pi/2$ (Problem 41) and at $x = \pi$ (Problem 42).]

41. $y'' + 4y = g(x),\ y(0) = 1,\ y'(0) = 2$, where

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2} \\ 0, & x > \frac{\pi}{2} \end{cases}$$

42. $y'' - 2y' + 10y = g(x),\ y(0) = 0,\ y'(0) = 0$, where

$$g(x) = \begin{cases} 20, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

Discussion Problems

43. Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c, and k are constants. The auxiliary equation of the associated homogeneous equation is

$$am^2 + bm + c = 0.$$

(a) If k is not a root of the auxiliary equation, show that we can find a particular solution of the form $y_p = Ae^{kx}$, where $A = 1/(ak^2 + bk + c)$.

(b) If k is a root of the auxiliary equation of multiplicity one, show that we can find a particular solution of the form $y_p = Axe^{kx}$, where $A = 1/(2ak + b)$. Explain how we know that $k \ne -b/(2a)$.

(c) If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y = Ax^2e^{kx}$, where $A = 1/(2a)$.

44. Discuss how the method of this section can be used to find a particular solution of $y'' + y = \sin x \cos 2x$. Carry out your idea.

45. Without solving, match a solution curve of $y'' + y = f(x)$ shown in the figure with one of the following functions:

(*i*) $f(x) = 1$, (*ii*) $f(x) = e^{-x}$,
(*iii*) $f(x) = e^x$, (*iv*) $f(x) = \sin 2x$,
(*v*) $f(x) = e^x \sin x$, (*vi*) $f(x) = \sin x$.

Briefly discuss your reasoning.

(a)

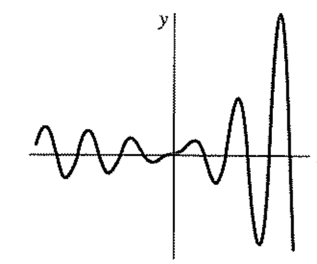

Figure 3.11 Solution curve

(b)

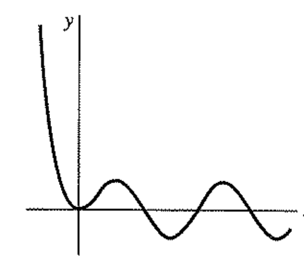

Figure 3.12 Solution curve

(c)

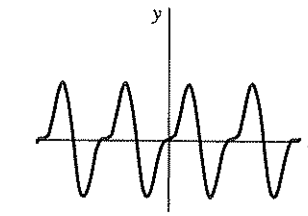

Figure 3.13 Solution curve

(d)

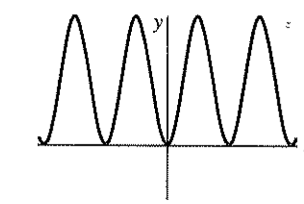

Figure 3.14 Solution curve

Computer Lab Assignments

In Problems 46 and 47, find a particular solution of the given differential equation. Use a CAS as an aid in carrying out differentiations, simplifications, and algebra.

46. $y'' - 4y' + 8y = (2x^2 - 3x)e^{2x} \cos 2x + (10x^2 - x - 1)e^{2x} \sin 2x$
47. $y^{(4)} + 2y'' + y = 2 \cos x - 3x \sin x$

3.5 Variation of Parameters

Introduction The **method of variation of parameters** used in Section 2.3 to find a particular solution of a linear first-order differential equation is applicable to linear higher-order equations as well. Variation of parameters has a distinct advantage over the method of the preceding section in that it *always* yields a particular solution y_p provided the associated homogeneous equation can be solved. In addition, the method presented in this section, unlike undetermined coefficients, is *not* limited to cases where the input function is a combination of the four types of functions listed on page 127, nor is it limited to differential equations with constant coefficients.

Some Assumptions To adapt the method of variation of parameters to a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{1}$$

we begin as we did in Section 3.2—we put (1) in the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing through by the lead coefficient $a_2(x)$. Equation (2) is the second-order analogue of the linear first-order equation $dy/dx + P(x)y = f(x)$. In (2) we shall assume $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I. As we have already seen we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$ and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

we find $\qquad u_1' = -\dfrac{2x^5e^x}{2x^3} = -x^2e^x \quad$ and $\quad u_2' = \dfrac{2x^3e^x}{2x^3} = e^x.$

The integral of the latter function is immediate, but in the case of u_1' we integrate by parts twice. The results are $u_1 = -x^2e^x + 2xe^x - 2e^x$ and $u_2 = e^x$. Hence

$$y_p = u_1y_1 + u_2y_2 = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x.$$

Finally we have $y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$. ❑

Remarks

The similarity between the forms of solutions of Cauchy-Euler equations and solutions of linear equations with constant coefficients is not just a coincidence. For example, when the roots of the auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1e^{m_1x} + c_2e^{m_2x} \quad \text{and} \quad y = c_1x^{m_1} + c_2x^{m_2}, \quad x > 0. \tag{5}$$

In view of the identity $e^{\ln x} = x$, $x > 0$, the second solution given in (5) can be expressed in the same form as the first solution:

$$y = c_1e^{m_1 \ln x} + c_2e^{m_2 \ln x} = c_1e^{m_1t} + c_2e^{m_2t},$$

where $t = \ln x$. This last result illustrates another fact of mathematical life: Any Cauchy-Euler equation can *always* be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The idea is to solve the new differential equation in terms of the variable t, using the methods of the previous sections, and once the general solution is obtained, resubstitute $t = \ln x$. Since this procedure provides a good review of the Chain Rule of differentiation, you are urged to work Problems 31–36 in Exercises 3.6.

EXERCISES 3.6

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18, solve the given differential equation.

1. $x^2y'' - 2y = 0$
2. $4x^2y'' + y = 0$
3. $xy'' + y' = 0$
4. $xy'' - 3y' = 0$
5. $x^2y'' + xy' + 4y = 0$
6. $x^2y'' + 5xy' + 3y = 0$
7. $x^2y'' - 3xy' - 2y = 0$
8. $x^2y'' + 3xy' - 4y = 0$
9. $25x^2y'' + 25xy' + y = 0$
10. $4x^2y'' + 4xy' - y = 0$
11. $x^2y'' + 5xy' + 4y = 0$
12. $x^2y'' + 8xy' + 6y = 0$
13. $3x^2y'' + 6xy' + y = 0$
14. $x^2y'' - 7xy' + 41y = 0$
15. $x^3y''' - 6y = 0$
16. $x^3y''' + xy' - y = 0$
17. $xy^{(4)} + 6y''' = 0$
18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

In Problems 19–24, solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$
20. $2x^2y'' + 5xy' + y = x^2 - x$
21. $x^2y'' - xy' + y = 2x$
22. $x^2y'' - 2xy' + 2y = x^4e^x$
23. $x^2y'' + xy' - y = \ln x$
24. $x^2y'' + xy' - y = \dfrac{1}{x+1}$

In Problems 25–30, solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0$, $y(1) = 0$, $y'(1) = 4$
26. $x^2y'' - 5xy' + 8y = 0$, $y(2) = 32$, $y'(2) = 0$
27. $x^2y'' + xy' + y = 0$, $y(1) = 1$, $y'(1) = 2$
28. $x^2y'' - 3xy' + 4y = 0$, $y(1) = 5$, $y'(1) = 3$
29. $xy'' + y' = x$, $y(1) = 1$, $y'(1) = -\frac{1}{2}$
30. $x^2y'' - 5xy' + 8y = 8x^6$, $y(\frac{1}{2}) = 0$, $y'(\frac{1}{2}) = 0$

In Problems 31–36, use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using the procedure in Sections 3.3–3.5.

31. $x^2y'' + 9xy' - 20y = 0$
32. $x^2y'' - 9xy' + 25y = 0$
33. $x^2y'' + 10xy' + 8y = x^2$
34. $x^2y'' - 4xy' + 6y = \ln x^2$
35. $x^2y'' - 3xy' + 13y = 4 + 3x$
36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

In Problems 37 and 38, solve the given initial-value problem on the interval $(-\infty, 0)$.

37. $4x^2y'' + y = 0$, $y(-1) = 2$, $y'(-1) = 4$
38. $x^2y'' - 4xy' + 6y = 0$, $y(-2) = 8$, $y'(-2) = 0$

Discussion Problems

39. How would you use the method of this section to solve
$$(x + 2)^2y'' + (x + 2)y' + y = 0?$$
Carry out your ideas. State an interval over which the solution is defined.
40. Can a Cauchy-Euler differential equation of lowest order with real coefficients be found if it is known that 2 and $1 - i$ are two roots of its auxiliary equation? Carry out your ideas.
41. The initial conditions $y(0) = y_0$, $y'(0) = y_1$, apply to each of the following differential equations:
$$x^2y'' = 0,$$
$$x^2y'' - 2xy' + 2y = 0,$$
$$x^2y'' - 4xy' + 6y = 0.$$
For what values of y_0 and y_1 does each initial-value problem have a solution?
42. What are the x-intercepts of the solution curve shown in Figure 3.15? How many x-intercepts are there in the interval $0 < x < \frac{1}{2}$?

Computer Lab Assignments

In Problems 43–46, solve the given differential equation by using a CAS to find the (approximate) roots of the auxiliary equation.

43. $2x^3y''' - 10.98x^2y'' + 8.5xy' + 1.3y = 0$
44. $x^3y''' + 4x^2y'' + 5xy' - 9y = 0$
45. $x^4y^{(4)} + 6x^3y''' + 3x^2y'' - 3xy' + 4y = 0$
46. $x^4y^{(4)} - 6x^3y''' + 33x^2y'' - 105xy' + 169y = 0$
47. Solve $x^3y''' - x^2y'' - 2xy' + 6y = x^2$ by variation of parameters. Use a CAS as an aid in computing roots of the auxiliary equation and the determinants given in (10) of Section 3.5.

3.7 Nonlinear Equations

Introduction The difficulties that surround higher-order *nonlinear* DEs and the few methods that yield analytic solutions are examined next.

Some Differences There are several significant differences between linear and nonlinear differential equations. We saw in Section 3.1 that homogeneous linear equations of order two or higher have the property that a linear combination of solutions is also a solution (Theorem 3.2). Nonlinear equations do not possess this property of superposability. For example, on the interval $(-\infty, \infty)$, $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cos x$, and $y_4 = \sin x$ are four linearly independent solutions of the nonlinear second-order differential equation $(y'')^2 - y^2 = 0$. But linear combinations such as $y = c_1e^x + c_3 \cos x$, $y = c_2e^{-x} + c_4 \sin x$, $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$ are not solutions of the equation for arbitrary nonzero constants c_i. See Problem 1 in Exercises 3.7.

In Chapter 2 we saw that we could solve a few nonlinear first-order differential equations by recognizing them as separable, exact, homogeneous, or perhaps Bernoulli equations. Even though the solutions of these equations were in the form of a one-parameter family, this family did not, as a rule, represent the general solution of the differential equation. On the other hand, by paying attention to certain continuity conditions, we obtained general solutions of linear first-order equations. Stated another way, nonlinear first-order differential equations can possess singular solutions whereas linear equations cannot. But the major difference between linear and nonlinear equations of order two or higher lies in the realm of solvability. Given a linear equation there is a chance that we can find some form of a solution that we can look at, an explicit solution or perhaps a solution in the form of an infinite series. On