and so on. Finally, we see that the general solution of the equation is $y = c_0 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

and $$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots.$$

Each series converges for all finite values of x. □

■ Nonpolynomial Coefficients The next example illustrates how to find a power series solution about the ordinary point $x_0 = 0$ of a differential equation when its coefficients are not polynomials. In this example we see an application of multiplication of two power series.

Example 5 ODE with Nonpolynomial Coefficients

Solve $y'' + (\cos x)y = 0$.

Solution We see $x = 0$ is an ordinary point of the equation because, as we have already seen, $\cos x$ is analytic at that point. Using the Maclaurin series for $\cos x$ given in (2), along with the usual assumption $y = \sum_{n=0}^{\infty} c_n x^n$ and the results in (1), we find

$$y'' + (\cos x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$= 2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \cdots = 0.$$

It follows that

$$2c_2 + c_0 = 0, \quad 6c_3 + c_1 = 0, \quad 12c_4 + c_2 - \frac{1}{2}c_0 = 0, \quad 20c_5 + c_3 - \frac{1}{2}c_1 = 0,$$

and so on. This gives $c_2 = -\frac{1}{2}c_0$, $c_3 = -\frac{1}{6}c_1$, $c_4 = \frac{1}{12}c_0$, $c_5 = \frac{1}{30}c_1, \ldots$. By grouping terms we arrive at the general solution $y = c_0 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots \quad \text{and} \quad y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots.$$

Since the differential equation has no finite singular points, both power series converge for $|x| < \infty$. □

■ Solution Curves The approximate graph of a power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ can be obtained in several ways. We can always resort to graphing the terms in the sequence of partial sums of the series; in other words, the graphs of the polynomials $S_N(x) = \sum_{n=0}^{N} c_n x^n$. For large values of N, $S_N(x)$ should give us an indication of the behavior of $y(x)$ near the ordinary point $x = 0$. We can also obtain an approximate solution curve by using a numerical solver as we did in Section 3.10. For example, if you carefully scrutinize the series solutions

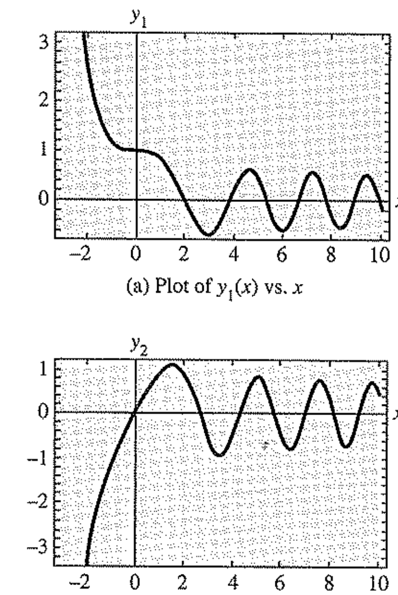
of Airy's equation in Example 2, you should see that $y_1(x)$ and $y_2(x)$ are, in turn, the solutions of the initial-value problems

$$y'' + xy = 0, \quad y(0) = 1, \quad y'(0) = 0,$$
$$y'' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1. \tag{11}$$

The specified initial conditions "pick out" the solutions $y_1(x)$ and $y_2(x)$ from $y = c_0 y_1(x) + c_1 y_2(x)$, since it should be apparent from our basic series assumption $y = \sum_{n=0}^{\infty} c_n x^n$ that $y(0) = c_0$ and $y'(0) = c_1$. Now if your numerical solver requires a system of equations, the substitution $y' = u$ in $y'' + xy = 0$ gives $y'' = u' = -xy$, and so a system of two first-order equations equivalent to Airy's equation is

$$y' = u$$
$$u' = -xy. \tag{12}$$

Initial conditions for the system in (12) are the two sets of initial conditions in (11) but rewritten as $y(0) = 1$, $u(0) = 0$, and $y(0) = 0$, $u(0) = 1$. The graphs of $y_1(x)$ and $y_2(x)$ shown in Figure 5.1 were obtained with the aid of a numerical solver using the fourth-order Runge-Kutta method with a step size of $h = 0.1$.



(a) Plot of $y_1(x)$ vs. x

(b) Plot of $y_2(x)$ vs. x

Figure 5.1 Solutions of Airy's equation

Remarks

(*i*) In the problems that follow, do not expect to be able to write a solution in terms of summation notation in each case. Even though we can generate as many terms as desired in a series solution $y = \sum_{n=0}^{\infty} c_n x^n$ either through the use of a recurrence relation or, as in Example 6, by multiplication, it may not be possible to deduce any general term for the coefficients c_n. We may have to settle, as we did in Examples 5 and 6, for just writing out the first few terms of the series.

(*ii*) A point x_0 is an ordinary point of a *nonhomogeneous* linear second-order DE $y'' + P(x)y' + Q(x)y = f(x)$ if $P(x)$, $Q(x)$, and $f(x)$ are analytic at x_0. Moreover, Theorem 5.1 extends to such DEs—in other words, we can find power series solutions $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ of nonhomogeneous linear DEs in the same manner as in Examples 2–5. See Problem 36 in Exercises 5.1.

EXERCISES 5.1

Answers to selected odd-numbered problems begin on page ANS-11.

5.1.1 Review of Power Series

In Problems 1–4, find the radius of convergence and interval of convergence for the given power series.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$
2. $\sum_{n=0}^{\infty} \frac{(100)^n}{n!}(x + 7)^n$
3. $\sum_{k=1}^{\infty} \frac{(-1)^k}{10^k}(x - 5)^k$
4. $\sum_{k=0}^{\infty} k!(x - 1)^k$

In Problems 5 and 6, the given function is analytic at $x = 0$. Find the first four terms of a power series in x. Perform the multiplication by hand or use a CAS, as instructed.

5. $\sin x \cos x$
6. $e^{-x}\cos x$

In Problems 7 and 8, the given function is analytic at $x = 0$. Find the first four terms of a power series in x. Perform the long division by hand or use a CAS, as instructed. Give the open interval of convergence.

7. $\frac{1}{\cos x}$
8. $\frac{1 - x}{2 + x}$

In Problems 9 and 10, rewrite the given power series so that its general term involves x^k.

9. $\sum_{n=1}^{\infty} nc_n x^{n+2}$
10. $\sum_{n=3}^{\infty} (2n - 1)c_n x^{n-3}$

In Problems 11 and 12, rewrite the given expression as a single power series whose general term involves x^k.

11. $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$

12. $\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$

In Problems 13 and 14, verify by direct substitution that the given power series is a particular solution of the indicated differential equation.

13. $y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (x+1)y'' + y' = 0$

14. $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \quad xy'' + y' + xy = 0$

5.1.2 Power Series Solutions

In Problems 15 and 16, without actually solving the given differential equation, find a lower bound for the radius of convergence of power series solutions about the ordinary point $x = 0$. About the ordinary point $x = 1$.

15. $(x^2 - 25)y'' + 2xy' + y = 0$
16. $(x^2 - 2x + 10)y'' + xy' - 4y = 0$

In Problems 17–28, find two power series solutions of the given differential equation about the ordinary point $x = 0$.

17. $y'' - xy = 0$
18. $y'' + x^2y = 0$
19. $y'' - 2xy' + y = 0$
20. $y'' - xy' + 2y = 0$
21. $y'' + x^2y' + xy = 0$
22. $y'' + 2xy' + 2y = 0$
23. $(x - 1)y'' + y' = 0$
24. $(x + 2)y'' + xy' - y = 0$
25. $y'' - (x + 1)y' - y = 0$
26. $(x^2 + 1)y'' - 6y = 0$
27. $(x^2 + 2)y'' + 3xy' - y = 0$
28. $(x^2 - 1)y'' + xy' - y = 0$

In Problems 29–32, use the power series method to solve the given initial-value problem.

29. $(x - 1)y'' - xy' + y = 0, \ y(0) = -2, \ y'(0) = 6$
30. $(x + 1)y'' - (2 - x)y' + y = 0, \ y(0) = 2, \ y'(0) = -1$
31. $y'' - 2xy' + 8y = 0, \ y(0) = 3, \ y'(0) = 0$
32. $(x^2 + 1)y'' + 2xy' = 0, \ y(0) = 0, \ y'(0) = 1$

In Problems 33 and 34, use the procedure in Example 6 to find two power series solutions of the given differential equation about the ordinary point $x = 0$.

33. $y'' + (\sin x)y = 0$
34. $y'' + e^x y' - y = 0$

Discussion Problems

35. Without actually solving the differential equation $(\cos x)y'' + y' + 5y = 0$, find a lower bound for the radius of convergence of power series solutions about $x = 0$. About $x = 1$.
36. How can the method described in this section be used to find a power series solution of the *nonhomogeneous* equation $y'' - xy = 1$ about the ordinary point $x = 0$? Of $y'' - 4xy' - 4y = e^x$? Carry out your ideas by solving both DEs.
37. Is $x = 0$ an ordinary or a singular point of the differential equation $xy'' + (\sin x)y = 0$? Defend your answer with sound mathematics.
38. For purposes of this problem, ignore the graphs given in Figure 5.1. If Airy's DE is written as $y'' = -xy$, what can we say about the shape of a solution curve if $x > 0$ and $y > 0$? If $x > 0$ and $y < 0$?

Computer Lab Assignments

39. (a) Find two power series solutions for $y'' + xy' + y = 0$ and express the solutions $y_1(x)$ and $y_2(x)$ in terms of summation notation.
    (b) Use a CAS to graph the partial sums $S_N(x)$ for $y_1(x)$. Use $N = 2, 3, 5, 6, 8, 10$. Repeat using the partial sums $S_N(x)$ for $y_2(x)$.
    (c) Compare the graphs obtained in part (b) with the curve obtained using a numerical solver. Use the initial-conditions $y_1(0) = 1$, $y_1'(0) = 0$, and $y_2(0) = 0$, $y_2'(0) = 1$.
    (d) Reexamine the solution $y_1(x)$ in part (a). Express this series as an elementary function. Then use (5) of Section 3.2 to find a second solution of the equation. Verify that this second solution is the same as the power series solution $y_2(x)$.
40. (a) Find one more nonzero term for each of the solutions $y_1(x)$ and $y_2(x)$ in Example 6.
    (b) Find a series solution $y(x)$ of the initial-value problem $y'' + (\cos x)y = 0$, $y(0) = 1$, $y'(0) = 1$.
    (c) Use a CAS to graph the partial sums $S_N(x)$ for the solution $y(x)$ in part (b). Use $N = 2, 3, 4, 5, 6, 7$.
    (d) Compare the graphs obtained in part (c) with the curve obtained using a numerical solver for the initial-value problem in part (b).

5.2 Solutions about Singular Points

Introduction The two differential equations $y'' + xy = 0$ and $xy'' + y = 0$ are similar only in that they are both examples of simple linear second-order DEs with variable coefficients. That is all they have in common. Since $x = 0$ is an *ordinary point* of the first equation, we saw in the preceding section that there was no problem in finding two distinct power series solutions centered at that point. In contrast, because $x = 0$ is a *singular point* of the second DE, finding two infinite series solutions—notice we did not say "power series solutions"—of the equation about that point becomes a more difficult task.

A Definition A singular point $x = x_0$ of a linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is further classified as either regular or irregular. The classification again depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

DEFINITION 5.2 Regular/Irregular Singular Points

A singular point x_0 is said to be a **regular singular point** of the differential equation (1) if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

The second sentence in Definition 5.2 indicates that if one or both of the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ fails to be analytic at x_0, then x_0 is an irregular singular point.

Polynomial Coefficients As in Section 5.1, we are mainly interested in linear equations (1) where the coefficients $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials with no common factors. We have already seen that if $a_2(x_0) = 0$, then $x = x_0$ is a singular point of (1) since at least one of the rational functions $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$ in the standard form (2) fails to be analytic at that point. But since $a_2(x)$ is a polynomial and x_0 is one of its zeros, it follows from the Factor Theorem of algebra that $x - x_0$ is a factor of $a_2(x)$. This means that after $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are reduced to lowest terms, the factor $x - x_0$ must remain, to some positive integer power, in one or both denominators. Now suppose that $x = x_0$ is a singular point of (1) but that both the functions defined by the products $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are analytic at x_0. We are led to the conclusion that multiplying $P(x)$ by $x - x_0$ and $Q(x)$ by $(x - x_0)^2$ has the effect (through cancellation) that $x - x_0$ no longer appears in either denominator. We can now determine whether x_0 is regular by a quick visual check of denominators: If $x - x_0$ appears *at most* to the first power in the denominator of $P(x)$ and *at most* to the second power in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point. Moreover, observe that if $x = x_0$ is a regular singular point and we multiply (2) by $(x - x_0)^2$, then the original DE can be put into the form

$$(x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0, \tag{3}$$

where p and q are analytic at $x = x_0$.

Example 1 Classification of Singular Points

It should be clear that $x = 2$ and $x = -2$ are singular points of

$$(x^2 - 4)^2y'' + 3(x - 2)y' + 5y = 0.$$

quotient can be carried out by hand. But all these operations can be done with relative ease with the help of a CAS. We give the results:

Here is a good place to use a computer algebra system.

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}\,dx = y_1(x)\int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right]^2}$$

$$= y_1(x)\int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots\right]} \quad \leftarrow \text{after squaring}$$

$$= y_1(x)\int \left[\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots\right] dx \quad \leftarrow \text{after long division}$$

$$= y_1(x)\left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right] \quad \leftarrow \text{after integrating}$$

or

$$y_2(x) = y_1(x)\ln x + y_1(x)\left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right].$$

On the interval $(0, \infty)$, the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$. ◻

Remarks

(*i*) The three different forms of a linear second-order differential equation in (1), (2), and (3) were used to discuss various theoretical concepts. But on a practical level, when it comes to actually solving a differential equation using the method of Frobenius, it is advisable to work with the form of the DE given in (1).

(*ii*) When the difference of indicial roots $r_1 - r_2$ is a positive integer ($r_1 > r_2$), it sometimes pays to iterate the recurrence relation using the smaller root r_2 first. See Problems 31 and 32 in Exercises 5.2.

(*iii*) Since an indicial r is a root of a quadratic equation, it could be complex. We shall not, however, investigate this case.

(*iv*) If $x = 0$ is an irregular singular point, we may not be able to find *any* solution of the DE of form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

EXERCISES 5.2

Answers to selected odd-numbered problems begin on page ANS-11.

In Problems 1–10, determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

1. $x^3y'' + 4x^2y' + 3y = 0$
2. $x(x+3)^2y'' - y = 0$
3. $(x^2 - 9)^2y'' + (x+3)y' + 2y = 0$
4. $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$
5. $(x^3 + 4x)y'' - 2xy' + 6y = 0$
6. $x^2(x-5)^2y'' + 4xy' + (x^2 - 25)y = 0$
7. $(x^2 + x - 6)y'' + (x+3)y' + (x-2)y = 0$
8. $x(x^2+1)^2y'' + y = 0$
9. $x^3(x^2 - 25)(x-2)^2y'' + 3x(x-2)y' + 7(x+5)y = 0$
10. $(x^3 - 2x^2 + 3x)^2y'' + x(x-3)^2y' - (x+1)y = 0$

In Problems 11 and 12, put the given differential equation into the form (3) for each regular singular point of the equation. Identify the functions $p(x)$ and $q(x)$.

11. $(x^2 - 1)y'' + 5(x+1)y' + (x^2 - x)y = 0$
12. $xy'' + (x+3)y' + 7x^2y = 0$

In Problems 13 and 14, $x = 0$ is a regular singular point of the given differential equation. Use the general form of the indicial equation in (14) to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius.

13. $x^2y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$
14. $xy'' + y' + 10y = 0$

In Problems 15–24, $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent series solutions about $x = 0$. Form the general solution on $(0, \infty)$.

15. $2xy'' - y' + 2y = 0$
16. $2xy'' + 5y' + xy = 0$
17. $4xy'' + \frac{1}{2}y' + y = 0$
18. $2x^2y'' - xy' + (x^2 + 1)y = 0$
19. $3xy'' + (2 - x)y' - y = 0$
20. $x^2y'' - (x - \frac{2}{9})y = 0$
21. $2xy'' - (3 + 2x)y' + y = 0$
22. $x^2y'' + xy' + (x^2 - \frac{4}{9})y = 0$
23. $9x^2y'' + 9x^2y' + 2y = 0$
24. $2x^2y'' + 3xy' + (2x - 1)y = 0$

In Problems 25–30, $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about $x = 0$. Use (21) where necessary and a CAS, if instructed, to find a second solution. Form the general solution on $(0, \infty)$.

25. $xy'' + 2y' - xy = 0$
26. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$
27. $xy'' - xy' + y = 0$
28. $y'' + \frac{3}{x}y' - 2y = 0$
29. $xy'' + (1 - x)y' - y = 0$
30. $xy'' + y' + y = 0$

In Problems 31 and 32, $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the recurrence relation found by the method of Frobenius first with the largest root r_1. How many solutions did you find? Next use the recurrence relation with the smaller root r_2. How many solutions did you find?

31. $xy'' + (x - 6)y' - 3y = 0$
32. $x(x-1)y'' + 3y' - 2y = 0$
33. (a) The differential equation $x^4y'' + \lambda y = 0$ has an irregular singular point at $x = 0$. Show that the substitution $t = 1/x$ yields the differential equation
$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \lambda y = 0,$$
which now has a regular singular point at $t = 0$.
(b) Use the method of this section to find two series solutions of the second equation in part (a) about the singular point $t = 0$.
(c) Express each series solution of the original equation in terms of elementary functions.
34. **Buckling of a Tapered Column** In Example 3 of Section 3.9, we saw that when a constant vertical compressive force or load P was applied to a thin column of uniform cross-section, the deflection $y(x)$ satisfied the boundary-value problem
$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$
The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compressive force is a critical load P_n.
(a) In this problem let us assume that the column is of length L, is hinged at both ends, has circular cross-sections, and is tapered as shown in Figure 5.2(a). If the column, a truncated cone, has a linear taper $y = cx$ as shown in cross section in Figure 5.2(b), the moment of inertia of a cross section with respect to an axis perpendicular to the xy-plane is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence we can write $I(x) = I_0(x/b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation in (24), we see that the deflection in this case is determined from the BVP
$$x^4\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(a) = 0, \quad y(b) = 0,$$
where $\lambda = Pb^4/EI_0$. Use the results of Problem 33 to find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.
(b) Use a CAS to plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b = 11$ and $a = 1$.

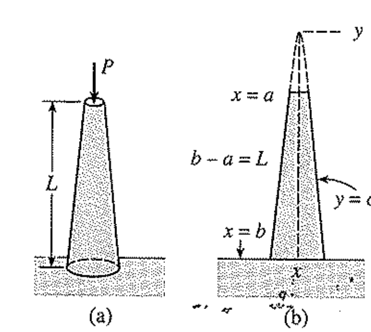


Figure 5.2 Tapered column in Problem 34